# TEST PAPERS IN GEOMETRY

FOR THE USE OF CANDIDATES PREPARING FOR SCHOOL CERTIFICATE, MATRICULATION AND SIMILAR EXAMINATIONS

BY

W. E. PATERSON, M.A., B.Sc.

FORMERLY EXAMINER FOR THE SCHOOL CERTIFICATE AND MATRICULATION EXAMINATIONS OF THE UNIVERSITIES OF LONDON DURHAM AND BRISTOL

LONDON
SIR ISAAC PITMAN & SONS, LTD.
PITMAN HOUSE, PARKER ST., KINGSWAY, W.C.2
BATH MELBOURNE
1935

# PREFACE

This collection of papers is intended to provide tests of the same standard as school certificate examinations, with questions of the same varied nature. They are not graded, and will probably be found to vary in difficulty just as actual examination papers vary from year to year.

Book work is included in the papers, partly to provide the necessary revision, but also to give some clue to the solution of the following rider.

It is advised that care and attention be given to questions requiring figures drawn to scale; facility in the accurate use of mathematical instruments is particularly important for the examination candidate who has no great skill in solving riders. When the figure is not required to be drawn to scale, it wastes time to use instruments. Figures for propositions and riders should be drawn freehand, but should be neat, of a fair size, and reasonably accurate.

The first fifty papers are limited to the syllabus of the Matriculation and School Certificate Examinations of the University of London; to meet the requirements of other examinations the last fifty papers include questions involving proportion and similarity, and questions needing elementary trigonometry. Further practice in trigonometrical manipulation can be obtained by solving scale-drawing questions by calculation.

A special edition is published, which gives the answers to numerical questions and skeleton solutions to all but the simplest riders. Proofs are given of many pieces of book work which are included in some textbooks of geometry but not in all.

W. E. PATERSON.

# TEST PAPERS IN GEOMETRY

## No. 1

1. In a triangle $ABC$ the angle $ABC$ is $120°$; equilateral triangles $DAB$, $EBC$ are described on the sides $AB$ and $BC$, so that they fall entirely outside the triangle $ABC$. Prove that $DE = AC$.

2. Define a parallelogram and prove that the diagonals bisect one another.

State and prove additional properties of the diagonals of (i) a rectangle, (ii) a rhombus.

3. Construct a triangle $ABC$, having $AB = 4{\cdot}7$ cm., $AC = 3{\cdot}4$ cm., and angle $BAC = 75°$. Draw the perpendicular bisector of $BC$ and the bisector of the angle $BAC$. Call their point of intersection $D$, and measure the angle $BDC$.

4. State and prove the geometrical proposition corresponding to the algebraical formula for $(a + b)^2$.

In a triangle $ABC$ with a right angle at $C$, a perpendicular $CD$ is let fall on $AB$. Prove that $CD^2 = AD \,.\, DB$.

5. Two straight lines $AX$, $AY$ contain an angle of $55°$, and a circle of radius 3 cm. rolls along $XA$ towards $A$.

Find by an accurate drawing the distance of the centre of the circle from $A$ when the circle touches $AY$. Give your reasons.

6. A circle is drawn having the side $BC$ of an acute-angled triangle $ABC$ as diameter. Prove that $A$ must be outside the circle.

If the circle cuts $AB$ at $P$, and $AC$ at $Q$, and if $BQ$, $CP$ cut at $X$, prove that $AX$ is perpendicular to $BC$.

## No 2

1 If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to one pair of equal sides equal, show that the angles opposite the other pair of equal sides are either equal or supplementary

If the bisector of the angle $A$ of a triangle $ABC$ also bisects the side $BC$, prove that $AB = AC$

2 If two non parallel lines are cut by a transversal, prove, without using any properties of parallel lines, that the two interior angles on that side of the transversal on which the given lines tend to meet are together less than two right angles

State the converse of this

✓3 Draw a triangle $ABC$, being given that $AB = 6 2$ cm, $AC = 5 7$ cm, and $AD$, the perpendicular from $A$ to $BC$, $= 3 8$ cm Give reasons for your construction and show that there are two solutions

4 If a straight line be divided into two parts, show that the sum of the squares on the whole line and one part is equal to twice the rectangle contained by the whole line and that part together with the square on the other part

In a square $ABCD$ $A$ is joined to any point $E$ in $BC$, prove that $AE^2 = 2 BC \cdot BE + CE^2$

5 Prove that the perpendicular from the centre of any circle on any chord of the circle bisects the chord

Take a point $A$ 1 7 in from the centre of a circle of radius 2 in Through $A$ draw a chord $BC$, such that $BC$ is bisected at $A$, also draw through $A$ a chord $PQ$ of length 3 in Prove your constructions

6 A statue $BC$ of height 8 ft stands on a pedestal $AB$ of height 11 ft, a man whose eye is 5 ft from the ground walks along a path which is at right angles to $AB$ Find two positions of his eye $E$ such that the angle $BEC = 20°$, and find, by measurement, the distance between the positions (Take 1 cm to represent 1 ft)

## No. 3

1. Two triangles have their angles equal, each to each, but they have no sides equal; prove that the small triangle may be placed so that two of its sides fall on two sides of the large triangle, and that the third sides are then parallel.

2. What are the respective loci of a point when it moves under the following conditions—

(i) So that it is always 6 in. from a fixed point;

(ii) So that it is always 6 in. from a fixed straight line;

(iii) So that the lines joining it to two fixed points always contain an angle 90°?

3. Draw a parallelogram with diagonals 2·3 in. and 1·5 in. long, and one side 1·1 in. long. Measure the obtuse angle contained by the diagonals.

4. Prove Apollonius's theorem, viz.—

The sum of the squares on two sides of a triangle is equal to twice the square on half the third side together, with twice the square on the median bisecting that side.

Two points $A$ and $B$ are 2 in. apart, draw the locus of a point $P$ which moves so that $PA^2 + PB^2 = 6\frac{1}{2}$ sq. in.

5. Prove that the radius of the inscribed circle of a triangle is equal to the area of the triangle divided by its semi-perimeter.

What is the diameter of the largest circular pond that can be made in a triangular field whose sides are 65 yds., 56 yds., 33 yds.?

6. From a point $P$ a tangent $PA$ is drawn to a circle and a secant $PBC$; from the secant $PD$ is cut equal to $PA$. Prove that $AD$ bisects the angle $BAC$.

## No. 4

1 The bisector of the opposite angles $A$ and $C$ of a parallelogram $ABCD$ meet the diagonal $BD$ at $E$ and $F$ respectively Prove that $AE = CF$

2 Two roads $AB$, $AC$ cross at an angle of $75°$, $B$ a mile stone is 800 yds from $A$ It is required to place a flagstaff equidistant from $A$ and $B$ and equidistant from the two roads By drawing a figure to scale find the distance of the flagstaff from $A$

3 Prove that the straight line joining the middle points of two sides of a triangle is parallel to the third side

If the middle points of the consecutive sides of any quadrilateral are joined, show that the quadrilateral so formed is always a parallelogram and that it is equal to half the original quadrilateral

4 From the right angle $A$ of a triangle $ABC$ a perpendicular $AD$ is let fall on the hypotenuse $BC$, prove that the square on $AB$ is equal to the rectangle $BC, BD$

Without any calculation, construct an isosceles right angled triangle equal in area to a rectangle 3 in by 1 3 in

5 If two circles cut, prove that the line joining their centres bisects the common chord at right angles

Word this enunciation so as to state a property of two isosceles triangles having a common base

6 Points $X$, $Y$, $Z$ are taken on a circle so that $YXZ$ is an acute angle and the minor arcs $XY$ $XZ$ are bisected at $P$ and $Q$ respectively The straight line $PQ$ cuts $XY$, $XZ$ in $R$ and $S$ Prove that $XR = XS$

## No. 5

1. If a triangle has two equal sides, prove that the angles opposite those sides are equal.

Discuss the proposition—

If a triangle has two sides which are nearly equal, then the angles opposite those sides are also nearly equal.

2. In a triangle $ACB$ the angle $ACB$ is obtuse. Prove that the perpendicular from $A$ on $BC$ falls outside the triangle.

In what well-known proposition is this fact assumed without proof?

3. Prove geometrically that if a parallelogram and a triangle are on the same base and between the same parallels, the area of the parallelogram is twice that of the triangle.

Describe an equilateral triangle with side 1 in., and then construct a parallelogram equal to the triangle in area, and having the longer sides three times the length of the shorter sides.

4. Two parallel chords of a circle of radius 6·5 in. are respectively 12·6 in. and 12 in. long. If the centre of the circle is between them, what is the distance between the chords?

If the two chords had been at right angles, calculate the distance of their point of intersection from the centre.

5. What is the definition of *touching* circles? Prove that if two circles touch, their centres and point of contact are in a straight line.

If two circles touch at $A$, and any line through $A$ meets one circle at $P$ and the other at $Q$, prove that the radii drawn to $P$ and $Q$ are parallel.

6. Two circles touch at $A$; through $A$ lines $PAQ$, $XAY$ are drawn, meeting one circle at $P$ and $X$, and the other at $Q$ and $Y$. Prove that $PX$ and $QY$ are parallel.

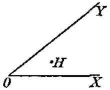
## No. 6

1 Correct the following enunciations—

(a) Two triangles are congruent if they have two sides and an angle of one respectively equal to two sides, and an angle of the other

(b) Two triangles are congruent if they have the angles of the one equal to the angles of the other, each to each

2 Define a parallelogram, and from the definition prove that the diagonals of a parallelogram bisect one another




$OX$, $OY$ represent straight portions of two railway lines, $H$ represents a house which is known to be equi-distant from two stations, one on each railway line, and in the same straight line with them State how the positions of the railway stations may be found If the figure is drawn to the scale of 1 cm to the mile, what is the distance between the stations ?

3 Construct a triangle $ABC$ of area 5 1 sq cm, having $AB = 3\ 4$ cm, $BC = 3\ 7$ cm

Measure the difference between the two possible values of the angle $B$

4 Divide a straight line $AB$ into two parts at $C$ so that the square on $AC$ is equal to twice the square on $BC$, and give a geometrical proof that the construction is correct

5 If a circle can be inscribed in a quadrilateral $ABCD$, prove that $AB + CD = AD + BC$

Construct such a quadrilateral, being given that the radius of the inscribed circle is 5 cm angle $A$ is 70°, angle $C$ is 80° $AB = 12$ cm Verify that $AB + CD = AD + BC$

6 An arc $AB$ of a circle is bisected at $D$, and $D$ is joined to the other extremity $C$ of the diameter $AC$ If $CD$ cuts $AB$ at $E$ prove that rect $DE$ $DC$ = sq on $DA$

## No. 7

1. Draw a line parallel to a given line $AB$ through any given point $P$ with ruler and compasses only. Prove your construction to be correct.

2. Define a rhombus, and from your definition prove that the diagonals of a rhombus bisect one another at right angles. Show that if a rhombus can be inscribed in a circle it must be a square.

3. In a triangle $ABC$ the perpendicular from $B$ on the bisector of the angle $C$ meets $AC$ (produced if necessary) at $D$. Prove that $AD$ is equal to the difference between $AC$ and $BC$.

4. If $D$ is the middle point of the side $BC$ of a triangle $ABC$, prove that—

$$AB^2 + AC^2 = 2BD^2 + 2AD^2.$$

Prove that, if the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral must be a parallelogram.

5. Prove that the greatest line that can be drawn from a point inside a circle to the circumference is that which passes through the centre.

$A$ and $B$ are two points within a circle, not on the same diameter. Find the point $P$ on the circumference such that $AP^2 + BP^2$ is as great as possible. Prove your construction to be correct.

6. If at a point a tangent and a chord are drawn, prove that the angles between the tangent and chord are equal to the angles in the alternate segments of the circle.

At a point $C$ on a circle a tangent is drawn which meets any chord $AB$, produced, at $D$. Perpendiculars $DE$ and $DF$ are let fall on $CB$ and $AC$, produced. Prove that $EF$ is at right angles to $AD$.

## No 8

1 The side $BC$ of a triangle $ABC$ is bisected at $D$, and $AD$ is produced to $E$ so that $AE = 2AD$ Prove that $ABEC$ is a parallelogram

Construct a triangle $ABC$ having $AB = 2$ in, $AC = 3$ in, and $AD$, the median bisecting $BC$, $= 2\ 3$ in Prove the truth of your construction

2 In the side $BC$ of a triangle $ABC$ any point $P$ is taken, show how to draw a line $PQ$, cutting $BA$ in $Q$, so that the triangle $PBQ$ is equal in area to the triangle $ABC$

3 If a straight line is divided into two equal and also two unequal parts, prove geometrically that the rectangle contained by the unequal parts together with the square on the line between the points of section is equal to the square on half the line

State the equivalent algebraical formula

4 Prove fully that angles at the circumference of a circle standing on the same chord are either equal or supplementary

$P$ is any point on the circumference of a circle of which $AB$ is a fixed chord The bisectors of the angles $PAB$, $PBA$ meet at $I$, find the locus of $I$

5 A line $XY$ is drawn 6 8 cm from the centre of a given circle of radius 4 cm. Draw a circle of radius 3 5 cm. to touch $XY$ and the given circle State the steps of your construction and prove your construction to be correct

6 Two straight lines $AD$, $BC$ meet at $O$ when produced Prove that, if rectangle $OA\ OD$ = rectangle $OB\ OC$, the angle $ABD$ = the angle $ACD$

## No. 9

1. The sides $AB$, $CB$ of a triangle $ABC$ are produced to $P$ and $Q$ respectively so that $BP = AB$ and $BQ = CB$. Prove that $PQ$ is equal and parallel to $AC$. Give the enunciation of all the propositions used in the proofs.

2. Prove that parallelograms on the same base and between the same parallels are equal in area.

A point $X$ is taken in the side $AB$ of a parallelogram $ABCD$; prove that the triangle $CXD$ equals the sum of the triangles $BXD$ and $AXD$.

3. A straight line $PO$ meets another straight line $XY$ at $O$, and the angles $POX$, $POY$ are bisected. From any point $A$ on the bisector of $POX$ a perpendicular $AB$ is drawn to $OP$, and produced to meet the bisector of $POY$ at $C$. Prove that the square on $AO$ is equal to the rectangle $AB \,.\, AC$.

4. In a triangle $ABC$, $AB = 3{\cdot}9$ cm., $BC = 4{\cdot}8$ cm., and the angle $BAC = 65°$; construct a rectangle equal in area to the triangle $ABC$, having one side of length $6{\cdot}5$ cm.

5. Give the construction, with proof, for describing a circle about a given triangle.

Two triangles have a side of one equal to a side of the other, and the angles opposite those sides equal; prove that the radii of their circumcircles are equal.

6. A circle is described having as diameter a radius $OA$ of a circle with centre $O$. From $A$ a straight line is drawn cutting the inner circle at $P$ and the outer circle at $Q$. Prove that $AQ$ is bisected at $P$.

Also if $PR$ is at right angles to $AO$ and is produced to meet the outer circle at $S$, prove that $AS^2 = 2AP^2$.

## No 10

1 A picture is supported by a string 4 ft long which is attached to two rings in the top of the picture frame, 2 ft apart The string passes over a nail in the wall and the top of the picture frame is horizontal If the string is shortened by 1 ft, find, by drawing a figure to scale, by how much the picture will be raised

2 A transversal cuts two parallel lines at $A$ and $B$ The two interior angles at $A$ are bisected and so are the two interior angles at $B$, the four bisectors forming a quadrilateral $ACBD$ Prove that (i) $ACBD$ is a rectangle, (ii) $CD$ is parallel to the original parallel lines

3 If the square on one side of a triangle is equal to the sum of the other two sides, prove that the triangle is right angled

The middle points of the sides $BC$, $CA$, $AB$ of a triangle are $P$, $Q$, $R$ respectively, and $AP$, $BQ$, $CR$ intersect at $G$ If $AP = 24$, $BQ = 30$ $CR = 18$, prove that the angle $PGC$ is a right angle

4 Show that four circles can be drawn so that each of them touches each of three given intersecting straight lines

5 Two lines $AX$, $AY$ meet at an angle of $43^\circ$, along $AX$ two points $B$ and $C$ are taken so that $AB = 1{\cdot}7$ in, and $AC = 2{\cdot}5$ in Construct a circle that shall pass through $B$ and $C$ and have its centre 1 in from $AY$ Show that there are two such circles

6 Any three points $A$, $B$, $C$ are taken on the circumference of a circle, $AD$ is drawn at right angles to $AB$ to meet the circle at $D$, and $CE$ is drawn at right angles to $CA$ to meet the circle at $E$ Prove that $DE$ is equal to $AB$

## No. 11

1. Prove that the angles at the base of an isosceles triangle are equal.

The base angles $B$ and $C$ of an isosceles triangle $ABC$ are bisected by lines meeting at $D$. Prove that $AD$ bisects the angle $A$.

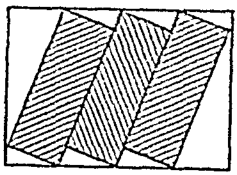
2. In what kinds of parallelograms—
(i) Are the diagonals at right angles?
(ii) Do the diagonals bisect the angles?
(iii) Are the diagonals equal?

Lines are drawn joining the middle points of adjacent sides of a rhombus. Prove that the quadrilateral so formed is a rectangle.

3. Prove that triangles on the same base with their vertices on a line parallel to the base are equal in area.

Draw a triangle $ABC$ having $AB = 4{\cdot}7$ cm., $BC = 3{\cdot}6$ cm., $CA = 5{\cdot}2$ cm.; and then construct a triangle equal to $ABC$ in area and having one side 7 cm. long, and another 3 cm. long.

4. The diagram represents three books, too tall for a bookshelf, which just fit in the shelf when inclined at 20° to the vertical. If the books are each 9 in. tall and 1·5 in. thick, find, by drawing to a scale 1 cm. = 1 in., the length and height of the shelf.



5. Draw two circles of radius ·7 in. to touch a given circle of radius 1 in., and to pass through a given point ·5 in. from the centre. Measure the distance between the centres of the two circles.

6. If in two circles equal chords subtend equal angles at points on the circumference, prove that the circles are equal.

In a triangle $ABC$ the sides $AB$, $AC$ are equal, a point $P$ on the other side of the base $BC$ is such that the angles $APB$, $APC$ are equal and $AP$ is not perpendicular to $BC$. Prove that the triangles $APB$, $APC$ have the same circumscribing circle.

## No. 12

1 In a certain pentagon the five sides are equal and the bisectors of the interior angles meet in a point Prove that all the angles of the pentagon are equal to one another, and that a circle can be described to pass through the middle points of the sides of the pentagon

2 A rectangle $ABCD$ has $AB$ of length 3 5 in, and $BC$ of length 2 in Construct geometrically a rectangle $ACEF$ of equal area and prove the construction to be correct

3 By a geometrical construction prove that the rectangle contained by the sum and difference of two straight lines is equal to the difference of the squares described on them

Draw a figure to illustrate the algebraical identity—

$$(a+b)^2-(a-b)^2=4ab$$

4 Explain what you understand by " a proof by *reductio ad absurdum* "

Write out the enunciation and proof of any proposition in which this method is used

5 If two chords of a circle intersect within a circle prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other

Two lines $AB$ $CD$ intersect at $O$, so that $AO\ OB = CO\ OD$, prove that the sum of the angles $ACB$ and $ADB$ is two right angles

6 The sides $BA$ and $CA$ of a triangle $ABC$ are produced to $D$ and $E$ respectively, making $AD$ equal to $CA$ and $AE$ equal to $BA$ The bisector of the angle $BAC$ meets $BC$ in $H$, $DE$ in $K$, and the circle circumscribing the triangle $ABC$ in $L$ Prove that—

(i) $B, L, D,$ and $K$ are concyclic

(ii) Rectangle $AH\ AL$ = rectangle $BA\ AC$

(iii) The square on $AH$ = rectangle $BA\ AC$ – rectangle $BH\ HC$

# AUTHOR'S NOTE

THESE "Points Essential to Answers" contain skeleton solutions to all the riders in the Test Papers ; they must be used sensibly or the student will lose rather than gain by their use. The student should make every effort to answer questions before consulting the solutions and must recollect that they are not model solutions but merely outlines. When the solution is understood, the student is advised to draw a fresh figure with different letters, and then to write out a fuller solution such as is required in an examination.

Riders frequently admit of several methods of solution. If a student finds that his solution differs from that given here, he is advised to revise it carefully, making sure that he has used all the data, has not inadvertently assumed the required result in the proof, and that he can give a reason for every step in the argument. If he is still satisfied with his proof, and, as is very probable, finds it shorter than the book solution, he will not be more pleased than will

W. E. PATERSON.

# TEST PAPERS IN GEOMETRY
# POINTS ESSENTIAL TO ANSWERS

## No 1

1 Since all angles at $B = 360°$, angle $DBE = 120°$, and triangles $ABC$ $DBE$ are congruent.

3 105°

4 $AC^2 + BC^2 = AD^2 + DC^2 + BD^2 + DC^2$
$AB^2 \quad = AD^2 + BD^2 + 2AD \quad DB$
but $AB^2 \quad = AC^2 + BC^2 \qquad CD^2 - AD \quad DB$

5 When circle touches $AY$ $AX$ and $AY$ are tangents and centre is on bisector of $XAY$ During rolling centre moves along line parallel to $XA$ at distance 3 cm Final d stance from $A$ — 6 5 cm

6. Acute angle $A$ is in segment larger than semi-circle. $A$ is outside semi circle Angles $BPC$ $BQC$ in semi-circle are r ght angles. $BQ$ $CP$ are two perpendiculars from vertices. Hence $AX$ is the third perpendicular as the three are concurrent

## No 2

1 By first part angles at $B$ and $C$ are e ther equal or supplementary Being angles of a triangle, they are not supplementary they are equal $AB = AC$

3 Draw $AD$ and erect perpendicular at $D$ on both sides. With centre $A$ radius 6 2 draw circle cutting perpendicular on one s de at $C$ With centre $A$ radius 5 7 draw circle cutting perpendicular on both sides Hence there are two non congruent tr angles $ABC$

4 $AE^2 = AB^2 + BE^2 = BC^2 + BE^2 = 2BC \quad BE + CE^2$

5 Join $A$ to centre $O$ and draw chord $BC$ at right angles to $OA$ Draw any chord $PQ = 3$ in With centre $O$ and radius the perpendicular from $O$ to $PQ$ draw circle from $A$ draw tangent to this circle and produce both ways cutting first circle at $D$ and $E$ Chords $DE$ and $PQ$ are equi distant from $O$ and therefore are equal i e $DE = 3$ in.

6 Draw horizontal $XY$ 6 ft from ground. On $BC$ draw segment to contain an angle 20° the arc cuts $XY$ at the two pos t ons of $E$ Distance $= 12\ 2$ ft

## No 3

3 Draw triangle $ABE$ having $AB = 1\ 1$ $AE = 1\ 15$ $BE = \ 75$ Then $E$ is intersection of diagonals and $AB$ is one s de Obtuse angle $= 113°$

4 If $O$ is mid point of $AB$ $PA^2 + PB^2 = 2AO^2 + 2PO^2$ $PO = 1\ 5$ in and locus is circle with centre $O$ radius 1 5 in.

5 If $I$ is centre triangles $BIC$ $CIB$ $AIB$ make up triangle $BIC$ $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$ — area $r$ = area — semi perimeter $65^2 - 56^2 = 121 \times 9 = 33^2$ triangle is right-angled Area $= 28 \times 33$ diameter $= 24$ yd.

6 $DAC + DCA = PDA = PAD = PAB + BAD$ But $PAB = DCA$ $DAC = BAD$

## No 4

1 Triangles $AED$ $BCF$ are congruent 1 side and 2 angles.

2 Flagstaff is on perpendicular bisector of $AB$ and on b sector of angle $BAC$ Distance $= 504$ yd.

3. Each joining line = half a diagonal and is parallel to diagonal. ∴ figure is parallelogram. Each triangle at corner is $\frac{1}{4}$ of a triangle cut off by diagonal. ∴ 4 triangles = half the parallelogram.

4. Make square = rect. 3 by 2·6 and draw diagonal.

6. Join $PZ$, $QY$

$$\begin{aligned} YRS &= QPY + PYX = YPZ + QPZ + PZX \\ &= X + \tfrac{1}{2}Y + \tfrac{1}{2}Z \\ ZSR &= PQZ + QZX = YQZ + PQY + QYX \\ &= X + \tfrac{1}{2}Z + \tfrac{1}{2}Y \end{aligned}$$

$\therefore XRS = XSR. \quad \therefore XR = XS.$

## No. 5

1. If a triangle has two sides which are nearly equal, then the angles opposite are either nearly equal or nearly supplementary.

2. Assume perpendicular falls inside and use *reductio ad absurdum*.

3. Bisect base $BC$ of triangle at $D$. With centre $B$, radius 1·5, describe circle to cut parallel through $A$ to $BC$ at $E$. Complete parallelogram $BEFD$.

4. 4·1 in. Distance from centre

$= \sqrt{\text{sum of square of distances of chords from centre}} = \sqrt{8{\cdot}81} = 2{\cdot}97.$

5. Two isosceles triangles have angles at base equal. ∴ angles at vertices equal. ∴ radii parallel.

6. Use angle between tangent and chord = angle in alternate segment.

## No. 6

2. Produce $OH$ to $K$ so that $OH = HK$. Through $K$ draw parallels to $OX$ and $OY$ meeting $OX$ at $A$, $OY$ at $B$. Then $A$ and $B$ are the stations = 5·2 miles.

3. Draw $XY$ parallel to $AB$ at distance 3 cm. With centre $B$, radius 3·7, describe circle. This cuts $XY$ in two places. ∴ two solutions. Difference = 71° 40′.

5. Let two tangents from $A$ be of length $x$, from $By$, from $Cz$, from $Dw$. Then $AB + CD = x + y + z + w$ and $AD + BC = x + y + z + w$.

Draw circle centre $O$, radius 5 cm., and any radius $OP$. Make angle $POQ = 110°$. Draw tangents at $P$ and $Q$ to meet at $A$. Produce $AP$ to $B$ so that $AB = 12$ cm. Draw $BR$ the other tangent from $B$. Make $ROS = 100°$. At $S$ draw tangent to meet $AQ$ at $D$ and $BR$ at $C$.

6. $DAE = DCB = DCA$. ∴ $DA$ is tangent to circle $AEC$. ∴ $DE \cdot DC = DA^2$.

## No. 7

2. Opposite angles equal and also supplementary. ∴ figure is a square.

3. If the perpendicular from $B$ on bisector meets bisector at $E$, then triangles $CED$, $BED$ are congruent and $CD = CB$.

4. Let $O$ be mid-point of diagonal $BD$ of the quadrilateral $ABCD$. Then $AB^2 + AD^2 + BC^2 + CD^2 = 2AO^2 + 2BO^2 + 2CO^2 + 2BO^2 = BD^2 + 2AO^2 + 2OC^2$. Hence $AC^2 = 2AO^2 + 2OC^2 = 4AP^2 + 4PO^2$, where $P$ is mid-point of $AC$. ∴ $PO^2 = 0$, i.e. $P$ and $O$ coincide, i.e. diagonals bisect one another. ∴ quadrilateral is a parallelogram.

5. Bisect $AB$ at $C$; then $AP^2 + BP^2 = 2AC^2 + 2CP^2$ ∴ $AP^2 + BP^2$ is greatest when $CP$ is greatest, i.e. when $CP$ passes through the centre.

6. Since $DFC + DEC = 2$ right angles. ∴ $CEDF$ is cyclic. ∴ $AFE = CDE$. Also $CD$ is a tangent. ∴ $FAB = DCE$. ∴ $FAB + AFE = CDE + DCE$ = 1 right angle.

# TEST PAPERS IN GEOMETRY
# POINTS ESSENTIAL TO ANSWERS

## No. 1

1. Since all angles at $B = 360°$, angle $DBE = 120°$, and triangles $ABC$ $DBE$ are congruent

3 105°.

4 $AC^2 + BC^2 = AD^2 + DC^2 + BD^2 + DC^2$
$AB^2 \quad = AD^2 + BD^2 + 2AD \cdot DB$
but $AB^2 \quad = AC^2 + BC^2 \quad \therefore CD^2 = AD \cdot DB$

5 When circle touches $AY$, $AX$ and $AY$ are tangents and centre is on bisector of $XAY$. During rolling, centre moves along line parallel to $XA$ at distance 3 cm. Final distance from $A = 6.5$ cm

6 Acute angle $A$ is in segment larger than semi circle $\therefore A$ is outside semi circle. Angles $BPC$, $BQC$, in semi circle, are right angles $\therefore BQ$, $CP$ are two perpendiculars from vertices. Hence $AX$ is the third perpendicular, as the three are concurrent

## No. 2

1 By first part, angles at $B$ and $C$ are either equal or supplementary. Being angles of a triangle, they are not supplementary $\therefore$ they are equal $AB = AC$

3 Draw $AD$ and erect perpendicular at $D$ on both sides. With centre $A$, radius 6.2, draw circle cutting perpendicular on one side at $C$. With centre $A$, radius 5.7, draw circle cutting perpendicular on both sides. Hence there are two non-congruent triangles $ABC$

4 $AE^2 = AB^2 + BE^2 = BC^2 + BE^2 = 2BC \cdot BE + CE^2$

5. Join $A$ to centre $O$ and draw chord $BC$ at right angles to $OA$. Draw any chord $PQ = 3$ in. With centre $O$ and radius the perpendicular from $O$ to $PQ$, draw circle from $A$ draw tangent to this circle and produce both ways cutting first circle at $D$ and $E$. Chords $DE$ and $PQ$ are equi distant from $O$ and therefore are equal i e $DE = 3$ in.

6 Draw horizontal $XY$, 5 ft. from ground. On $BC$ draw segment to contain an angle 20°, the arc cuts $XY$ at the two positions of $E$. Distance = 12.2 ft

## No. 3

3. Draw triangle $ABE$ having $AB = 1.1$, $AE = 1.15$, $BE = .75$. Then $E$ is intersection of diagonals and $AB$ is one side. Obtuse angle = 113°

4 If $O$ is mid point of $AB$, $PA^2 + PB^2 = 2AO^2 + 2PO^2 \quad \therefore PO = 1.5$ in and locus is circle with centre $O$ radius 1.5 in

5 If $I$ is centre, triangles $BIC$, $CIB$, $AIB$ make up triangle $BIC$ $\therefore \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr =$ area. $\quad r =$ area $-$ semi perimeter $\quad 65^2 - 56^2 = 121 \times 9 = 33^2 \quad \therefore$ triangle is right angled. Area $= 28 \times 33$, diameter $= 24$ yd.

6 $DAC + DCA = PDA = PAD = PAB + BAD$. But $PAB = DCA$ $\therefore DAC = BAD$

## No. 4

1 Triangles $AED$, $BCF$ are congruent, 1 side and 2 angles.

2 Flagstaff is on perpendicular bisector of $AB$ and on bisector of angle $BAC$. Distance = 504 yd.

3. Each joining line = half a diagonal and is parallel to diagonal. ∴ figure is parallelogram. Each triangle at corner is $\frac{1}{4}$ of a triangle cut off by diagonal. ∴ 4 triangles = half the parallelogram.

4. Make square = rect. 3 by 2·6 and draw diagonal.

6. Join $PZ$, $QY$

$$\begin{aligned} YRS &= QPY + PYX = YPZ + QPZ + PZX \\ &= X + \tfrac{1}{2}Y + \tfrac{1}{2}Z \\ ZSR &= PQZ + QZX = YQZ + PQY + QYX \\ &= X + \tfrac{1}{2}Z + \tfrac{1}{2}Y \end{aligned}$$

$$\therefore XRS = XSR. \quad \therefore XR = XS.$$

## No. 5

1. If a triangle has two sides which are nearly equal, then the angles opposite are either nearly equal or nearly supplementary.

2. Assume perpendicular falls inside and use *reductio ad absurdum*.

3. Bisect base $BC$ of triangle at $D$. With centre $B$, radius 1·5, describe circle to cut parallel through $A$ to $BC$ at $E$. Complete parallelogram $BEFD$.

4. 4·1 in. Distance from centre

$= \sqrt{\text{sum of square of distances of chords from centre}} = \sqrt{8{\cdot}81} = 2{\cdot}97.$

5. Two isosceles triangles have angles at base equal. ∴ angles at vertices equal. ∴ radii parallel.

6. Use angle between tangent and chord = angle in alternate segment.

## No. 6

2. Produce $OH$ to $K$ so that $OH = HK$. Through $K$ draw parallels to $OX$ and $OY$ meeting $OX$ at $A$, $OY$ at $B$. Then $A$ and $B$ are the stations = 5·2 miles.

3. Draw $XY$ parallel to $AB$ at distance 3 cm. With centre $B$, radius 3·7, describe circle. This cuts $XY$ in two places. ∴ two solutions. Difference = 71° 40′.

5. Let two tangents from $A$ be of length $x$, from $B$ $y$, from $C$ $z$, from $D$ $w$. Then $AB + CD = x + y + z + w$ and $AD + BC = x + y + z + w$.

Draw circle centre $O$, radius 5 cm., and any radius $OP$. Make angle $POQ = 110°$. Draw tangents at $P$ and $Q$ to meet at $A$. Produce $AP$ to $B$ so that $AB = 12$ cm. Draw $BR$ the other tangent from $B$. Make $ROS = 100°$. At $S$ draw tangent to meet $AQ$ at $D$ and $BR$ at $C$.

6. $DAE = DCB = DCA$. ∴ $DA$ is tangent to circle $AEC$. ∴ $DE \cdot DC = DA^2$.

## No. 7

2. Opposite angles equal and also supplementary. ∴ figure is a square.

3. If the perpendicular from $B$ on bisector meets bisector at $E$, then triangles $CED$, $BED$ are congruent and $CD = CB$.

4. Let $O$ be mid-point of diagonal $BD$ of the quadrilateral $ABCD$. Then $AB^2 + AD^2 + BC^2 + CD^2 = 2AO^2 + 2BO^2 + 2CO^2 + 2BO^2 = BD^2 + 2AO^2 + 2OC^2$. Hence $AC^2 = 2AO^2 + 2OC^2 = 4AP^2 + 4PO^2$, where $P$ is mid-point of $AC$. ∴ $PO^2 = 0$, i.e. $P$ and $O$ coincide, i.e. diagonals bisect one another. ∴ quadrilateral is a parallelogram.

5. Bisect $AB$ at $C$; then $AP^2 + BP^2 = 2AC^2 + 2CP^2$ ∴ $AP^2 + BP^2$ is greatest when $CP$ is greatest, i.e. when $CP$ passes through the centre.

6. Since $DFC + DEC = 2$ right angles. ∴ $CEDF$ is cyclic. ∴ $AFE = CDE$. Also $CD$ is a tangent. ∴ $FAB = DCE$. ∴ $FAB + AFE = CDE + DCE$ = 1 right angle.

## No. 8

1 (i) Triangles $ABD$, $EDC$ are congruent, 2 sides and included angle

(ii) First construct triangle $ABE$ having $AB = 2$, $BE = 3$, $AE = 4\ 6$ Complete parallelogram $ABEC$, then $ABC$ is required triangle

2 Draw $CQ$ parallel to $PA$ to meet $BA$ at $Q$

4 If $P$ is in major arc, angle $APB$ is constant and acute, and angle $AIB = 90 + \frac{1}{2}APB$

If $P$ is in minor arc, angle $APB$ is constant and obtuse and angle $AIB = 90 + \frac{1}{2}APB$

$\therefore$ complete locus of $I$ is the arcs of two segments, one on each side of $AB$, that on the major side containing an angle $90 + \frac{x}{2}$, that on the minor containing $180 - \frac{x}{2}$ where $x°$ is the acute angle $APB$

5 Centre of required circle is on line parallel to $XY$ at distance 3 5 cm, and is also on circle centre $O$, radius 7 5

6 Because rectangles are equal, the points $A$, $B$, $C$, $D$ are concyclic.

## No. 9

2 Triangle $BXD$ and triangle $AXD = \triangle\ ADB = \frac{1}{2}$ parallelogram = triangle $CXD$

3 Bisectors of adjacent angles are at right angles triangle $AOC$ is right angled *with $OB$ perpendicular to hypotenuse $BC$ Hence, by proof of* Pythagoras, $AO^2 = AB \cdot AC$

4 Construct rectangle $ABPQ$ equal to triangle $ABC$, produce $AB$ to $D$, making $AD = 6{\cdot}5$ Draw $BF$ parallel to $DQ$, meeting $AQ$ at $F$, then $ADEF$ is required rectangle

5 Angles at centres subtended by the equal sides are equal hence the radii are equal.

6 Angle $APO$ is a right angle, being in semi circle $OP$ bisects $AQ$. If $AOB$ is diameter of large circle then $ASB$ is a right angled triangle with $SR$ perpendicular to hypotenuse $AB$ $AS^2 = AB\ AR = 2AO\ AR = 2AP^2$

## No. 10

1 61 of a foot = 7 3 in

2 Bisectors of adjacent angles are perpendicular angles at $A$ and $B$ are right angles Angles on same side of transversal together equal two right angles their halves are equal to one right angle angles $C$ and $D$ are right angles Again, diagonals of rectangle and bisect one another angle $DCA$ = angle $BAC$ = angle $CAX$ where $AX$ is the original parallel $AX$ parallel to $DC$

3 Produce $AP$ to $H$ so that $GP = PH$ It is known by proof of concurrency of medians that $GH = \frac{2}{3}AP$ $CG = \frac{2}{3}CR$ and $CH = \frac{2}{3}BQ$ But $BQ^2 = AP^2 + CR^2$ $CH^2 = GH^2 + CG^2$ and $CGH$ is a right angle

5 Centre must lie on right bisector of $BC$ and also on one of the lines parallel to $AY$ at distance 1 in. from it

6 $AE$ and $BD$ are diameters and bisect one another Hence $ABED$ is a parallelogram. $AB = DE$

## No. 11

1 Halves of base angles are equal. $DB = DC$, $\therefore$ triangles $ADB$, $ADC$ are congruent

2 Diagonals of rhombus are at right angles. Lines joining mid points of two sides of a triangle are parallel to the third side.

3. Draw $CX$ parallel to $BA$; with centre $B$, radius 7, describe circle cutting $CX$ at $D$. Then traingle $BAD$ = triangle $ABC$. Through $A$ draw $AY$ parallel to $BD$, with centre $D$, radius 3, describe circle cutting $AY$ at $E$. Triangle $BDE$ = triangle $BDA$.

4. Length, 7·68 in.; height, 8·97 in.

5. With radius ·3 and centre as radius, describe circle, with point as centre and radius ·7, describe circle. Points of intersection are required centres. Distance is 0·52 of an inch.

6. Since $A$ is on the right bisector of $BC$ and $PA$ bisects angle $BPC$. $\therefore$ $ABCP$ are concyclic. $\therefore$ triangles $APB$, $APC$ have same circumcircle.

Or equal chords $AB$, $AC$ subtend equal angles at $P$, $\therefore$ circles are equal. Angles $ABP$, $ACP$ are not equal, $\therefore$ circles coincide.

## No. 12

1. Let $ABCDE$ be pentagon, and let $O$ be point where bisectors of angles meet. Then each adjacent pair of triangles are congruent, 2 sides and included angles. Hence all angles at bases are equal and angles of pentagon are equal. Also perpendiculars from $O$ bisect sides of pentagon and are equal.

2. Draw through $D$ a parallel to $AC$ and complete the rectangle with $AC$ as base.

5. For points $A$, $B$, $C$, $D$ are concyclic.

6. Triangles $BAC$, $EAD$ are congruent and $AK$ bisects angle $DAE$; hence $AK = AH$.

(i) $\therefore$ angle $BLK$ = angle $BCA$ = angle $BDK$, and $B, L, D, K$ are concyclic.

(ii) $\therefore$ rectangle $AH \,.\, AL$ = rectangle $AK \,.\, AL$ = rectangle $BA \,.\, AD$
= rectangle $BA \,.\, AC$.

but rectangle $AH \,.\, AL$ = rectangle $AH(AH + HL)$ = square on $AH$
+ rectangle $AH \,.\, HL$.

(iii) $\therefore$ square on $AH$ = rectangle $BA \,.\, AC$ − rectangle $AH \,.\, HL$
= rectangle $BA \,.\, AC$ − rectangle $BH \,.\, HC$.

## No. 13

1. Triangles $EBG$, $GCK$, $KDM$, $MAE$ are congruent.
$\therefore EG = GK = KM = ME$.
Also angle $EGB$ + angle $CGK$ = angle $EGB$ + angle $GEB$ = 1 right angle.
$\therefore$ angle $EGK$ is a right angle, similarly for the other angles.

2. Each triangle at corner $= \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{9}$ of the square.
$\therefore$ $EGKM = \frac{5}{9}$ of the square.

3. Let $AB = a$, $BC = b$, $CD = c$. Then
$$AB \,.\, CD + AD \,.\, BC = ac + b(a + b + c) = (a + b)(b + c) = AC \,.\, BD.$$

4. See Paper 5, No. 5, and use *reductio ad absurdum*.

5. The isosceles triangle is largest. Measurement = 1·89 sq. in.

6. $SQP = 90 - PQB = 90 - PAC = SRP$. $\therefore P, Q, R, S$ lie on a circle.
Also $CR \,.\, CS = CQ \,.\, CP = CA \,.\, CB$ = constant.

## No. 14

2. Angle $C$ > angle $B$ $\therefore C + \frac{1}{2}A > B + \frac{1}{2}A > \frac{1}{2}$ two right angles > right angle. $\therefore ADB = C + \frac{1}{2}A$ > a right angle.

3. Draw $AD$ = 4·8, cut off $AE$ = 3·2, at $E$ make $DEC$ = 42°, at $D$ make $EDC$ = 68°. Complete parallelogram $AECB$. Area = 4·4 sq. cm.

4. (i) Prove by *reductio ad absurdum*, viz., 2 angles of triangle = 2 right angles.

(ii) If circles cut in 3 points, 2 common chords both centres where right bisectors intersect circles would have equal radii, which is not true

5 Draw in same straight line $AB = 3\ 4$, $BC = 2\ 3$; from $B$ draw any line $BD = 2\ 8$ Draw circle $ADC$ cutting $DB$, produced at $E$ Measure $BE$

6 Angle $QPR$ = angle $PAB$ (in alternate segment) $= 90 - PBA = ORB = PRQ$

## No. 15

1 Let $ABCD$ be quadrilateral and $E$ intersection of diagonals

From 4 big triangles $ABC, BCD, CDA, DAB$ we have $AB + BC > AC$, etc, whence sum of sides > sum of diagonals

From 4 small triangles $ABE, BCE\ CDE, DAE$, we have $AE + EB > AB$, etc., whence twice sum of diagonals > sum of sides

2 Triangles $BAC$, $DAC$ are congruent Hence triangles $DAE$, $BAE$ are congruent

3 See Paper 81, No 3 Triangles $HEG$, $HFG$ are equal in area.

4 Place the two triangles with supplementary angles adjacent, they are then seen to have equal altitudes.

Triangle $AGE$ = triangle $ABC$, equal sides and included angles supplementary, etc

5 If small circle cuts $OA$ at $Q$ between $O$ and $P$ and $R$ between $P$ and $A$ then $OR$ is longest line from $O$ to small circle the chord perpendicular to $OR$ is shortest chord that touches small circle The diameters that touch small circle are the longest chords.

6 $PQ$ is chord subtending constant angle, radius of circumcircle is constant, etc

## No. 16

2 5 96 ft

3 Let $QP$ produced meet $BA$ produced at $R$ Angle $OQP$ = angle $OBR$ Therefore $PRB = QOB = POA = 45°$

5 Angles $BFC$, $BEC$ are right angles $B, F, E, C$ are concyclic, etc

6 Angle $PAC$ = angle $CBA$ (alternating segment) = angle $CAB$

$AD$ is at right angles to $CA$ and therefore bisects the angle supplementary to $PAB$

## No. 17

2 $P$ is intersection of right bisector of $AB$ and the bisector of angle $B$

3 Suppose $AD = a$, $BC = b$, and $a > b$ Draw $CE$ perpendicular to $AD$ $CD^2 = CE^2 + DE^2 = (a+b)^2 + (a-b)^2 = 2(a^2+b^2) = 2(AD^2 + BC^2)$

4 Angle $BAQ = 180 - BAP = \frac{1}{2}(360 - 2BAP) = \frac{1}{2}(360 - BOP) = \frac{1}{2}$ reflex angle $BOP$

5 On radius $OA$ as diameter describe circle, with centre $B$ and radius $BA$ describe circle cutting the last circle at $C$ Join $AC$ and produce to $P$, draw $BQ$ parallel to $AP$ $OCA$ is right angle in semi circle $AP = 2PC$

$AP$, $BQ$ are parallel chords, $PQ = AB = BC$, also angle $BAP$ = angle $QPA$ angle $BCA$ = angle $BAC$ = angle $QPA$ $BC$ is equal and parallel to $PQ$ $PC = BQ$ and $AP = 2BQ$

6 Triangles $DAF$, $EAF$ are congruent, two sides and included angle angle $AEF$ = angle $ADF$ = right angle Hence $A\ E, F, D$ lie on a circle Produce $HF$ right bisector of $AB$ to meet $DC$ at $G$ Then angle $FEG = 60°$ and triangle $EFG$ is half an equilateral triangle

$$\therefore DF = FE = 2GE = 2CD\left(1 - \frac{\sqrt{3}}{2}\right) = CD\,(2 - \sqrt{3})$$

## No. 18

1. Triangles $BAD$, $CAE$ are congruent, two sides and included angle.
2. Draw $PQ$ parallel to $XA$ meeting $AY$ at $Q$. On $QY$ cut off $QB = QA$. Join $BP$ and produce to meet $AX$ at $C$. Then $BC$ is bisected at $P$. Or method of Paper 6, Question 2, can be used.
3. Join $BE$. Triangle $BEF = \frac{1}{2}$ triangle $ADE$, triangle $BEC = \frac{1}{2}$ parallelogram $ABCE$. $\therefore$ triangle $BFC = \frac{1}{2}$ trapezium $ABCD$.
4. Both angles are equal to $AED$.
5. Let $P$ be point of contact of third tangent. $CO$ bisects angle $ACD$ and $DO$ bisects angle $BDC$, and these angles are supplementary. $\therefore COD$ is a right angle. $\therefore OP^2 = CP \cdot PD$ (Paper 1, Question 4) $= AC \cdot BD$.
6. $BQ^2 = BP \cdot BA = AP^2$. $\therefore BQ = BR = AP$. $\therefore AR = BP$.

## No. 19

1. Triangles $ADB$, $FEB$ are congruent. $\therefore$ angle $ABD$ = angle $FBE$.
$\therefore$ angles $ABD + FBD$ = angles $FBE + FBD$ = 2 right angles, etc.
2. Angles $EDB$, $AEB$ are obtuse. Hence $DE < BE$ (triangle $BDE$) $< AB$ (triangle $ABE$). Place triangles so that right angles coincide, then $A$ falls between $F$ and $D$; hence $B$ must fall beyond $E$. $\therefore$ angle $ABC$ is less than angle $DEF$.
3. Measurement = 6·7 sq. cm.
4. If $AD$ is median, and $AE$ the perpendicular on $BC$, then
$AB^2 - AC^2 = BE^2 - EC^2 = (BE + EC)((BE - EC) = 2BC \cdot DE$.
5. Centres of both arcs lie on the perpendicular to $PQ$ at $Z$. $XY$ = 7·15.
6. Draw right angled triangle having hypotenuse $PQ$, and $PR$ = 1 in. Required line is parallel to $QR$ at distance 2 in., on side remote from $P$.

## No. 20

1. $x = 2y$. $\therefore y = 36$, $x = 72$. $AD = DB > DC$.
2. Let fall $AD$ perpendicular to $XY$, produce to $E$, making $AD = DE$ Join $BE$ cutting $XY$ at $C$.
3. Draw $DE$ perpendicular to $AC$ and therefore parallel to $BC$. Then $CE = \frac{1}{2}CA$, etc.
4. $IB$, $EB$ bisect adjacent angle. $\therefore IBE$ is a right angle; so also is $ICE$.
5 Triangles $ACD$, $BCE$ are congruent, two sides and included angle. $\therefore$ angle $CBE$ = angle $DAC$. Hence, angle $BFD$ = angle $ACB$ = angle $ABC$. $\therefore$ Circle $BDF$ touches $AB$.

## No. 21

2. Locus is a line $XY$ parallel to base $AB$ at distance 4 cm., such that $AXYB$ is a rectangle. Triangle of minimum perimeter is the isosceles triangle $CAB$, with $AB$ as base. Produce $BY$ to $D$, making $YD = BY$. Take any point $P$ in $XY$. Then $CB = CD$ and $PB = PD$. Now $AP + PD > AD$. $\therefore AP + PB > AC + CB$.
3. Draw $AB$ = 4·3; mark off $AP$ = 3, $PQ$ = 1. On $AQ$ as diameter draw semi-circle cutting $PR$ (perpendicular to $AQ$) at $R$. On $AB$ as diameter draw semi-circle, through $R$ draw parallel to $AB$ meeting this semi-circle at $S$. Draw $SC$ perpendicular to $AB$. Then
$$AC \cdot CB = CS^2 = PR^2 = AP \cdot PQ = 3 \text{ sq. cm.}$$
4. Radius = 3·3 cm.

5 If $B$ and $E$ are on same side of $AD$, then angle $ABD$ = angle $AED$ If $B$ and $E$ are not on same side of $AD$, then angle $AED$ + angle $ACD = 180°$

6 Bisect $AB$ at $C$, $A$ and $B$ being the centres of the circles. Draw $QPR$ perpendicular to $PC$ For proof, let fall perpendicular on $QR$ from $A$ and $B$

## No. 22

1 Distance 95 1 yd.

2 Angles at point $= 5 \times 180° - 10$ angles at bases
$= 5 \times 180° -$ twice exterior angles of pentagon
$= 5 \times 180° - 2 \times 4$ right angles $= 180°$

3 Diagonal $AC$ cuts $BD$ at $H$ and $EF$ at $G$ Since $F$ and $E$ are mid points of $CD$ and $CE$, $FE$ is parallel to $BD$ and bisects $HC$ ∴ $AG = 3GC$, and triangle $AFG = 3$ triangle $CFG$ ∴ triangle $AFE = 3$ triangle $CFE$ Also triangle $CFE = \frac{1}{4}$ triangle $CBD = \frac{1}{8}$ parallelogram ∴ triangle $AFE$ + triangle $CFE = \frac{1}{2}$ parallelogram.

4 *Draw any chord of length 1 9 and mid point $Q$ Draw concentric circle* with radius $OQ$ From $P$ draw tangent to this circle cutting outer circle at $A$ and $B$

5 Produce $AB$, joining the two points to meet line at $P$ Determine $PQ$ the side of square equal to rectangle $PA \cdot PB$, and along line mark off $PC$ equal to $PQ$ Circle $ABC$ is required circle, there are two solutions, as $PC$ can be marked off in either direction

6. Reverse the proof for making isosceles triangle with each of the angles at base double of the angle at the vertex

## No. 23

1 Draw line $AB$ and erect equilateral triangle $BAC$, draw $BAD$ perpendicular to $AB$ Bisect angle $CAD$

3 $1\ 7^2 - 1\ 3^2 = 3 \times\ 4 = 1\ 2 < 1\ 21$ ∴ $1\ 7^2 < 1\ 3^2 + 1\ 1^2$ ∴ triangle is acute angled and circumcentre is inside

4 (i) Bisect arc $AB$ at $C$, then angle $AOB = 2$ angle $AOC = 2$ angle $POQ$
(ii) Chord $AC$ = chord $BC$ ∴ chord $AC > \frac{1}{2}$ chord $AB >$ chord $PQ$ ∴ angle $AOC$ is not equal to angle $POQ$ Hence angle $AOB$ is not double angle $POQ$

5 Since $ABD = ACD$, quadrilateral is cyclic ∴ $BE \cdot ED = AE \cdot EC$ ∴ $ED = 3$ Draw $AC = 3\ 5$ cm and cut off $AE = 2$ cm On $AE$ describe segment to contain an angle 100°, in it draw chord $EB = 1$ cm Produce $BE$ to $D$, making $ED = 3$ cm

6 (i) $AP = AQ$ $BP = BR$, $CQ = CR$ etc
(ii) $AX = AB - BX = AB - BZ$ $AY = AC - CX$, $AX = AY$

$$2AX = AB + AC - BC$$

(iii) $AC + CR = AC + CQ = AQ = AP = AB + BR$

$$AC - AB = BR - CR$$

$$AC - CZ = AC - CY = AY = AX = AB - BZ$$

$$AC - AB = CZ - BZ$$

$$2(AC - CB) = BR - BZ + CZ - CR = 2ZR$$

## No. 24

1 Angle $ABC$ = angle $ACB = 75°$, etc

2 Through $P$ draw a parallel to either diagonal of parallelogram $ABCD$

3 Let $p$ in, $q$ in, be lengths of perpendiculars from opposite vertices on diagonal of length 8 in Then area $= \frac{1}{2} \times 8 \times (p + q)$ ∴ $p + q = 6$ = other diagonal. ∴ the perpendiculars coincide with diagonals, i e diagonals are at *right angles Second part follows by Pythagoras*

4. Describe concentric circles with radii $(4 + 3)$, $(4 - 3)$, and $(5 + 3)$, $(5 - 3)$ respectively. The points of intersection give possible centres for the third circle.

5. Quadrilateral $PHAL$, $PKBL$, are each cyclic. Angle $PLH = PAH$ (same segment) $= PBA$ (alternate segment) $= PKL$ (same segment), etc.

6. Triangle is right angled. $\sqrt{5} = \sqrt{2^2 + 1^2}$. $\sqrt{7}$ = side of square = rectangle 7 by 1. A neat method is: Draw equilateral triangle $ABC$ with side 2 cm. Let fall $AD$ perpendicular to $BC$; produce $DC$ to $E$, making $DE = 2$ cm. With $D$ as centre, radius $DA$, describe circle cutting $AD$ produced at $F$; with $A$ as centre, radius $AE$, describe arc cutting previous circle at $G$. Triangle $AFG$ is required triangle.

## No. 25

1. $PR = 4{\cdot}8$ cm., $QR = 3{\cdot}6$ cm.

2. By book-work, $DE$ is parallel to $BA$ and $DF$ to $CA$. Hence $AEDF$ is a parallelogram and diagonal $AD$ is bisected by diagonal $FE$. Also triangle $DEF$ = each of the triangles $AFE$, $BDF$, $CED = \frac{1}{4}$ of triangle $ABC$.

3. See Paper 19, Question 4.

Or, if $AB > BC$, $ADB$ is obtuse and $AB^2 = BD^2 + AD^2 + 2BD \, . \, DX$

$ADC$ is acute and $AC^2 = CD^2 + AD^2 - 2CD \, . \, DX$

$\therefore AB^2 - AC^2 = 2BC \, . \, DX.$

4. Let fall $CD$ perpendicular to $AB$, then rectangle $AB \, . \, AD$ = square on $AC$.

5. Make $AOB = 40°$, $ABC = 95°$, $CD = CB$. Angle $BAD = 2 \times BAC = 130°$.

6. Let circles intersect at $AB$; the common chord $AB$ produced is locus, for if $P$ is any point on that line, rectangle $PA \, . \, PB$ = square on either tangent. If circles do not intersect, let $A$ and $B$ be centres and $R$, $r$ the radii.

Let $PQ$ be perpendicular to $AB$. Tangents are equal. $\therefore PA^2 - R^2 = PB^2 - r^2$.

$\therefore PA^2 - PB^2 = R^2 - r^2$. $\therefore AQ^2 - QB^2 = R^2 - r^2$.

$\therefore (AQ + QB)(AQ - QB) = R^2 - r^2$.

$\therefore AQ - QB = (R^2 - r^2) \div AB$ = constant.

$\therefore Q$ is a fixed point, and locus of $P$ is perpendicular to $AB$ at $Q$.

## No. 26

1. Three lengths must be measured.

2. Triangles $ABP$, $ACQ$ are congruent (1 side and 2 corresponding angles). $\therefore AP = AQ$ and hence angle $APQ$ = angle $ACB$. $\therefore PQ$ is parallel to $CB$.

3. Square $AF$ = parallelogram $ACHL$. (Same base, $AC$.)

But angle $HCF = 90 - ACH = ACB$. $\therefore$ triangles $ABC$, $FHC$ are congruent (1 side and 2 corresponding angles). $\therefore CH = CB = CD$. $\therefore$ parallelogram $ACHL$ = rectangle $CK$.

4. Angle $XCE$ = supplement of $BCX = XAB = XCD$ (since $XB = XD$).

5. $DA = DB = DC$ (tangent from external point). $\therefore$ angle $BAC$ is a right angle. $PD$ bisects $BDA$, $QD$ bisects $CDA$. $\therefore PDQ$ is a right angle.

6. Take any point $C$ on circumference and draw tangent $CD = AB$; with centre $O$ of circle as centre, and radius $OD$, draw circle to cut $AB$ produced at $P$, etc.

## No. 27

1. Angle $AEC$ = angle $ACE$. $\therefore$ angle $DEC$ = angle $BCE$.

Angle $ADB$ = angle $ABD$. $\therefore$ angle $BDE$ = angle $DBC$.

Hence angles $DEC$ and $BDE$ together equal half angles of quadrilateral $BDEC$. $\therefore CE$ and $BD$ are parallel.

2. 85°, 110°, 95°, 70°.

3. $17{\cdot}8^2 - 16^2 = 33{\cdot}8 \times 1{\cdot}8 = 169 \times {\cdot}36 = 7{\cdot}8^2$. $\therefore$ triangle is right angled. $\therefore$ Area $= \frac{1}{2} \times 16 \times 7{\cdot}8 = 62{\cdot}4$ sq. in.

4 Use first part twice, double, add

5. Draw perpendicular at $B$ Join $BC$ and produce to cut circle at $E$, join $EO$ and produce to cut perpendicular at $F$ $F$ is centre of one of required circles Join $BD$ to cut circle at $G$, produce $OG$ to meet perpendicular at $H$ $H$ is the other centre

6 Let $PEQ$ be tangent to circle $ABE$

Then angle $PED = BEQ = EAB$ (alternate segment) $= ECD$ $\therefore$ $PEQ$ is a tangent to circle $CED$

Let $RES$ be common tangent to circles $ADE$, $BCE$

Then angle $ADE = AER$ (between chord and tangent) $= SEC = EBC$ (alternate segment) $\therefore$ $AD$ parallel to $BC$ and $ABCD$ is a parallelogram

## No. 28

1 $OA = OA'$, $OB = OB'$, $AB = A'B'$ angle $AOB$ = angle $A'OB'$

2 $AP$ equal and parallel to $DQ$ $\therefore$ $PQ$ equal and parallel to $AD$ and $BC$ $\therefore$ the two parts of intercepted parallelogram equal the two parts $AQ$, $BQ$ of parallelogram $ABCD$

3 $BD^2 = 65^2 - 60^2 = 25^2$, $CD^2 = 156^2 - 60^2 = 144^2$, $BC = 169$ and $169^2 - 156^2 = 65^2$

5. Diameter is 10 cm Shortest chord is 8 cm

6 $O$ describes an arc of a circle, centre $C$, radius $CO$, intercepted between perpendicular to $AB$ at $A$ and perpendicular at same distance on other side of $BC$ $A$ describes arc of circle, centre $C$, radius $CA$, so that angle $ACA$ = angle $OCO$ $BA = 20$ in

## No 29

2 $PH$, $RL$, $QK$ are parallel and $PP = RQ$ $HL = LK$

$OL = OH + HL$ and $OL = OK - KL$ $2OL = OH + OK$

Through $R$ draw $SRT$ parallel to $OK$ meeting $PH$ at $S$, $KQ$ at $T$ (either produced, if necessary) Triangles $SPP$, $QRT$ are congruent, and $SP = TQ$ Hence $2RL = PH + QK$

3 Area $= 6.15$ sq cm

4 $AP^2 + BP^2 = 2AO^2 + 2PO^2$ $PO$ is constant

5 Quadrilateral $DBCE$ is cyclic

6 Draw $BC = 1$ in, and perpendicular $CD = \frac{1}{2}$ in

Centre $D$, radius $DC$, draw arc cutting $BD$, produced at $E$

Centre $B$, radius $BE$, draw circle

Centre $C$, radius $BE$, draw circle, cutting previous circle at $A$

$ABC$ is required triangle

Let $F$ be point where $B$ circle cuts $CB$ produced. Join $AF$

$$\begin{aligned} \text{Rectangle } CB \cdot CF &= CB \cdot BF + CB^2 = CB \cdot BF + BD^2 - DC^2, \\ &= CB \cdot BF + (BD + DE)(BD - DC) \\ &= CB \cdot BF + BF(BD - DC) = BF(2DE + BD - DE) \\ &= BF^2 \end{aligned}$$

$\therefore$ $CB \cdot CF = CA^2$ $CA$ touches circle $ABF$ angle $BAC$ = angle $BFA$, but angle $ABC$ (at centre) $= 2$ angle $BFA$ (at circumference) $ABC = 2BAC$ and $BAC = 36°$

## No. 30

1 By using Pythagoras, it is seen that $DE > CB$ $CE > CB$ angle $CAE > CAB$ See also Paper 19, Question 2

2 $DE$ is parallel to $BA$, equal to $\frac{1}{2}AB$ equal to $\frac{1}{2}AC$, $DK$ is $\frac{1}{2}EC$, but $DH$ is also $\frac{1}{2}EC$ $DH = DK$

But $HK$ is parallel to $DC$ and $AD$ is perpendicular to $BC$ $AD$ is perpendicular to $HK$, but $DH = DK$ $AD$ is the right bisector of $HK$ $AH = AK$

3. First make parallelogram equal to square, with sides 2 in. and $2\frac{1}{2}$ in. Then make rhombus with side $2\frac{1}{2}$ in. equal to parallelogram.

4. Draw triangle $ABC$ and about it describe circle. On $AB$ describe segment to contain angle 140°, arc cutting $AC$ at $O$. Produce $AO$ to cut circle at $D$. $CD = 2{\cdot}2$ in.

5. Draw any radius $OA$ and make $AOB = 160°$, let fall $OC$ perpendicular to $AB$ and draw a circle with centre $O$ and radius $OC$. From $P$ draw tangent to this circle, cutting former circle in $QR$. The minor segment cut off by $QR$ contains 100°.

6. Angle $DBK$ = angle $DAC$ = angle $EBD$. $\therefore$ triangles $DBK$, $DBE$ are congruent. $\therefore DK = DE$.

## No. 31

1. Join $EO$, $FO$; triangles $AOC$, $BOD$ are congruent. $\therefore AC = BD$. $\therefore AE = BF$ and triangles $EOA$, $BOF$ are congruent. $\therefore$ angle $EOA$ = angle $BOF$, etc. Or, $\therefore$ diagonals bisect one another, $ACBD$ is a parallelogram. $\therefore AE$ equal and parallel to $BF$, etc.

3. First make square equal to 20, i.e. 5 by 4 rectangle. Divide line 10 long so that rectangle contained by parts equals the square.

4. (i) Diagonals bisect. $\therefore$ quadrilateral is a parallelogram in a circle. $\therefore$ a rectangle. $\therefore$ diagonals are diameters.

(ii) $CD$ is bisected at $O$. $\therefore$ triangles $BOC$, $AOD$ are congruent and $AO = OB$.

5. Triangles $ABC$, $DBC$ congruent. $\therefore$ angle $BAC$ = angle $BDC$, etc.

Chord $AB$ = chord $CD$. $\therefore$ arc $AB$ = arc $CD$. $\therefore$ angle $ACB$ = angle $CAD$. $\therefore AD$ and $BC$ are parallel.

6. (i) Join $QB$. Angle $PQB = \frac{1}{2}ADB = 30°$, angle $QPB = ACB = 60°$. $\therefore QBR$ is 90° and $QR$ is diameter.

(ii) Angle $SRB = \frac{1}{2}ADB = 30°$, $SBR = 90°$. $\therefore ASB = 120°$, $AOB = 60°$. $\therefore S$ is on circle $ACB$.

## No. 32

1. 10·4 ft.

2. If $ABCDEF$ is hexagon, produce $BA$ to meet $EF$ at $G$.
Each angle of hexagon is 120°.
$\therefore$ angle $AGF = 120 - 60$. $\therefore 60$. $\therefore AGF + ABC = 180°$, etc.

3. Fig. $RACB$ is a parallelogram. $\therefore RA = BC$, similarly $AQ = BC$.
Hence perpendicular from $A$ on $BC$ is perpendicular bisector of $QR$. $\therefore$ etc.

4. $CO = CB$. $\therefore$ angle $OCA$ = 2 angle $BOC$; also angle $OAC = OBC = BOC$. $\therefore AOD$ = 3 times $BOC$.

5. Let $R$ be point of contact with $BC$. $AP = AB + BR$, $AQ = AC + CR$. $\therefore AP + AQ$ = perimeter. $\therefore AP = s$.

6. Length = 8·48 ft.

## No. 33

1. Triangles $PAC$, $QAB$ are congruent. $\therefore$ angles $PBX$ and $QCX$ are equal. Triangles $PBX$, $QCX$ are congruent. $\therefore BX = CX$. Triangles $BAX$, $CAX$ are congruent, etc.

2. Produce $AB$ to $E$, making $AE = CD = 4{\cdot}5$. Make $BC = 3{\cdot}6$, $EC = 3$; complete parallelogram $AECD$.

3. (i) Triangles $BAE$, $DAG$ are congruent (2 sides and included angle). $\therefore$ rectangles are equal.

(ii) In triangle $DAE$, $DE^2 = DA^2 + AE^2 - 2DA \cdot AH$, and
$DE^2 = DA^2 + AE^2 - 2AE \cdot AL$, etc.

4 Parallelogram inscribed in circle opposite angles equal and supplementary a rectangle Parallelogram described about circle opposite sides equal and one pair of opposite sides — other pair of opposite sides (Paper 5 Quest on 5) all sides equal

5 Let $AB$ be common chord then $PT^2 = PA \cdot PB = PQ \cdot PP'$ $PT$ touches circle $QRT$

6 $BC = 3\ 9''$ $CA = 4\ 44$ $AB = 3\ 11$

## No 34

1 Cut off $AE$ from $AB$ equal to $AC$ Triangles $ADE$ $ADC$ are congruent and $DE = DC$

Angle $BED > ADE > ADC$ and $ADC > EBD$

$BED > EBD$ $BD > DE > DC$

2 Draw perpendiculars $DBF$ and $ECG$ to $BC$ at $B$ and $C$ Draw two parallels to $BC$ at distance 2 cm one meeting the perpendiculars at $D$ and $E$ the other at $F$ and $G$ On $BC$ as diameter describe circle cutting $DE$ at $H$ and $K$ and $FG$ at $L$ and $M$ The parallels now read $DHKE$ $FLMG$ The complete locus is made up of the four finite lines $DH$ $KE$ $FL$ $MG$

3 $C$ is mid point of $EA$ and $CF$ parallel to $AB$ $F$ is mid point of $BE$ Also $DC = AB = 2CF$

4 Draw circle first in it place chord $BC = 2$ in etc There are two solutions

5 Prove $PC$ is a tangent to the circle $ACF$ Angle $PCA = ADE$ (parallel) $= ABC$ (same segment) etc

6 Quadrilateral $PQRS$ can be superposed on quadrilateral $XPQR$ $PS = XR$

Also from triangle $PAX$ $QBR$ $AX = BR$ $XR = AB$

Construction. From $AC$ cut off $AE = AB$ draw $ES$ parallel to $AB$ meeting $BC$ at $S$ etc

$2QB^2 = QR^2$ $2QB = QR\sqrt{2}$ $AP + PQ + QB = AB$ $x(1 + \sqrt{2}) = a$

## No 35

1 From triangle $AEC$ $DFB$ $AE = DF$ $AEFD$ is a parallelogram etc

2 Area $= 4\ 13$ sq in

3 Pentagon $= \frac{1}{2}[HF + LD + AH] = \frac{1}{2}[AB^2 + BC^2]$

$= \frac{1}{2}[AC^2 + 2BC^2 + 2AC \cdot CB] - \frac{1}{2}[AC^2 + CF^2] + AC \cdot CB$

5 Angle $B = 60°$ $CAD = ABD$ and $CA$ is tangent to circle $ABD$ etc

6 $AD$ bisects angle $BAC$ and so passes through $I$

Angle $DIC = IAC + ICA = \frac{1}{2}(A + C)$

Angle $DCI = DCB + ICB = DAB + ICB = \frac{1}{2}(A + C)$ etc

## No 36

2 Triangles $XAZ$ $YBX$ $ZCY$ are congruent $XYZ$ is equilateral

Let $P$ be mid point of $YC$ Then triangle $XBP$ is equilateral $XY$ is perpendicular to $BP$

3 First make parallelogram on base 5 cm equal to rectangle with one side 6 cm etc

4 Indirect proof Take $AD$ to bisect angle and meet circumference at $D$ Then arc $DB$ = arc $DC$ chord $DB$ = chord $DC$ $D$ is on perpendicular bisector of $BC$

5 $DE$ is common chord of two circles having as diameters $AB$ and $CK$ etc

6 Distance of $A$ from $D$ the mid point of $BC$ is $\frac{1}{2}AC$

## No. 37

1. $BDC = 56°$.
2. $DF$ is parallel to $CA$. $FA > PF$. $\therefore DQ > DP$.
3. Draw $BQ$ parallel to $PC$.
5. On $AB$ as diameter, describe semi-circle, in it place chord $BC = 1{\cdot}9$ in., etc.
6. Let common tangent at $A$ meet $PT$ at $S$.

Angle $RAT = APT + ATP$, but $ATP = TAS$ and $APT = QAS$.
$\therefore$ angle $RAT =$ angle $QAT$.

## No. 38

2. $BD$ and $CE$ are two medians of triangle $ACD$.
3. Distance $= \frac{1}{2}\sqrt{7}$ miles $= 1{\cdot}32$ miles.
4. Angle is $66°$.
5. Angle is $45°$.
6. Equate two values for $PA^2 + PB^2 + PC^2$.

$P$ lies on each circle.

## No. 39

1. House is centre of circumscribed circle.
2. $67\frac{1}{2}°$ and $135°$.
3. Angle $ACD$ is a right angle; also $PD = PB$.

Hence $AP^2 + PB^2 = AP^2 + PD^2 = AD^2 = AC^2 + CD^2$; etc.

4. Line joining any two of the centres and line joining the other two bisect adjacent angles and are therefore perpendicular.
5. $A' = 66°$, $B' = 59°$, $C' = 55°$.
6. Triangles $ABP$, $ACQ$ are congruent. $\therefore APB = AQX$. $\therefore A, P, Q, X$ are on a circle. So also are $A, B, C, X$ and $AX$ is common chord of two circles, etc.

## No. 40

1. Triangles $BCF$, $AED$ are congruent. $\therefore BF = DE$. Hence $AC$ bisects $EF$, etc.
2. If $AD$ is median, triangles $BDA$, $CDA$ are equal; so also are $BDG$ and $CDG$. $\therefore$ remainder $AGB =$ remainder $CGA$.
3. Draw $EF$ parallel to $DA$ to meet $BA$ produced at $H$; draw $CK$ parallel to $DB$ to meet $AB$ produced at $K$. $DHK$ is required triangle.
4. Triangles $AOC$, $BOD$ are congruent (2 sides and included angle).
5. $BFEC$ are concyclic. $\therefore$ angle $AEF =$ angle $B$. Similarly $DEC =$ angle $B$. $\therefore DEF = 180 - 2B$.

But $BC$ is diameter and $P$ the centre of circle $BFEC$. $\therefore$ angle $FPC = 2B$.
$\therefore FPD + DEF = 180°$, i.e. $D, E, F, P$ lie on circle.

6. $OP$ and $OQ$ are at right angles to $AD$ and $BC$.

Angle $AOC = 2$ angle $ABC$ and angle $BOD = 2$ angle $BCD$.
$\therefore AOC + BOD = 2$ right angles. $\therefore BOC + AOD = 2$ right angles.
$\therefore COQ + AOP = 1$ right angle $= AOP + PAO$. Hence $COQ = PAO$.
$\therefore$ triangles $COQ$, $AOP$ are congruent, and $OP = CQ$.

## No. 41

2. $D$ is intersection of bisector of exterior angles at $A$ and $C$.
3. Triangle $BCG =$ triangle $AEB$ (equal bases, same vertex).

Triangle $GBF =$ triangle $GDE$ (equal bases, same vertex)
$=$ triangle $DEC +$ triangle $DCG$.
$=$ triangle $ADC$, since triangle $DCA =$ triangle $ADE$.
$\therefore$ triangle $EFG = EBC + BCG + GBF =$ quadrilateral $ABCD$.

4 Loci are lines parallel to $AB$ at distance 1 25 cm
On $AB$ as diameter describe circle in it put chord $BC = 3$ 5 With centre $A$ radius $AC$, describe circle cutting loci at $P$ and $Q$ etc
5 See Paper 40, Question 5 which shows $XYC = ZYA$
6 Suppose $AC$ greater than $AB$ Cut off $AD = AB$, then $ABD$, $AQR$ are both equilateral

$AR$ $RC = PR$ $RS = PQ$ $RS + QR$ $RS$
$AR$ $RD = AQ$ $QB = PQ$ $QR + PQ$ $RS$
$AR(RC - RD) = QR(RS - PQ)$ $AC - AB = QS - PR$

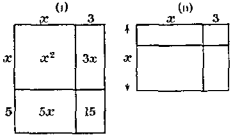
## No. 42

2 Triangles $ABF$, $CDF$ equal in area (equal bases between same parallels) etc
3 $AB^2 = AC^2 + BC^2 - 2BC$ $CD = AC^2 + BC$ $CD + BC$ $BD - 2BC$ $CD$
$= AC^2 + BC$ $CD$
4 Right bisectors of $AB$ and $BC$ meet at $P$
5 Draw $BC$ at $B$ make $CBI = 20°$, draw a parallel to $BC$ distant 1 in to meet $BI$ at $I$ Describe the inscribed circle with centre $I$ From $B$ and $C$ draw tangents to meet at $A$
6 Squares of tangents from point of intersection equal same rectangle

## No 43

1 Produce $BP$ to cut $AC$ at $D$
$BA + AC - BA + AD + DC > BD + DC > BP + PD + DC$
$> BP + PC$ Angle $BPC > PDC > BAC$
Neither converse is necessarily true
2 Angle $CAD$ = angle $BAD$ = angle $ADC$ $CD = CA$
If $BD$ were parallel to $AC$ figure would be parallelogram and $CD = AB$ Hence $AB$ would equal $AC$, which is known not to be true

3




$(x - 5)(x + 3)$
$= x^2 + 3x - 5x - 15$

4 $R$ mid point of hypotenuse $OR = \frac{1}{2}PQ$ = constant
5 Angle $ADC$ = angle $BAD$ arc $AC$ = arc $BD$ chord $AC$ = chord $BD$
Let common tangent $QPR$ cut $BC$ at $Q$
Angle $FEP = RPF$ (chord and tangent) $= BPQ = BAP$ $EF$ is parallel to $AD$
Angle $EPC = 180° - EDC = DEF - 180$ $FPD - BPD$
6 Median $AP = \frac{3}{2}AG = 3$ 6 Draw $AD$ and erect perpendicular at $D$ cutting circle centre $A$ and radius 3 6 at $P$ Draw perpendicular at $P$ to $PD$ cutting circle centre $A$ and radius 2 at $O$ the centre of the circumcircle etc

## No. 44

1. Triangles $ADF$, $BCG$ are congruent. $\therefore DF = CG$, and $EF = EG$.
2. Angle $EFG = 60° =$ angle $FDC$.
3. Make square equal to rectangle 7 by 3.
4. If $D$ is mid-point of $AB$, $PA^2 + PB^2 = 2AD^2 + 2PD^2$. $\therefore$ point $P$ is such that $DP$ is greatest.
5. Diagonals bisect adjacent angles and are, therefore, at right angles. By congruent triangles, the diagonals bisect one another. Hence quadrilateral is a rhombus.
6. Angle $PRQ$ is a right angle. $\therefore ABRP$ is a rectangle and $AP = BR$.
If $O$ is mid-point of $PQ$, perpendicular from $O$ on $AB$ bisects $AB$ and $XY$. $\therefore AX = BY$. $\therefore AX + AY = BY + AY = a$; also $BX . BY = AX . AY$.
But $BX . BY = BR . BQ = 1 \times b = b$. $\therefore AX . AY = b$. $\therefore AX$. $AY$ are roots of $x^2 - ax + b = 0$.

## No. 45

1. Point is intersection of diagonals of the parallelogram.
2. $BC = 2{\cdot}67$ cm.
3. $B, Y, X, C$ are concyclic. $\therefore AX . AC = AY . AB$.
$$BC^2 = BA^2 + AC^2 - 2BA . AY = BA^2 - BA . AY + AC^2 - CA \quad AX$$
$$= BA . BY + CA . CX.$$
4. Draw any transversal and bisect the four angles so formed. Line joining points of intersection of the bisectors would bisect the angle between original lines.
5. Produce $BA$ to $CD$ to meet at $R$; let $S$ denote semi-perimeter of triangle $PQR$. Then $SB = SC = S$, $SA = SD = S - PQ$. $\therefore AB = CD = PQ$. See Paper 23, Question 6.
6. Angles $ABR$, $ABS$ are right angles. $\therefore AP . AR = AB^2 = AQ . AS$, etc.

## No. 46

1. Mirrors are perpendicular bisectors of line joining point to images and of line joining successive images. $\therefore OP = OP_1 = OP_2$, etc.
2. Through $H$ draw $PHQ$ parallel to $AB$ and $XHY$ parallel to $AD$, meeting $AD$ in $P$, $BC$ in $Q$, $AB$ in $X$, $DC$ in $Y$.
Triangle $AHC$ + triangle $AHD$ + triangle $DHC$ = triangle $ADC$.
$\therefore 2AHC + 2AHD + 2DHC$ = parallelogram $ABCD$.
$\therefore 2AHC + 2AHD$ = parallelogram $ABCD$ − parallelogram $DPQC$.
= parallelogram $APQB$ = 2 triangle $AHB$.
$\therefore AHC = AHB - AHD$.
4. Place any chord $= 2{\cdot}0$ in circle, bisect it at $C$; draw concentric circle touching chord at $C$. From $P$ draw $PAB$ touching this circle.
5. Angle $OBA = OAB = ARB$ (alternate segment) $= AXO$ (parallel).
$\therefore X$ is on circle $OAB$, which passes through $C$, the centre of original circle.
$\therefore OXC = OAC$ = right angle, so $X$ is mid-point of $PQ$.
6. Let $O$ be centre of circle; on $OC$ as diameter describe circle, cutting $AB$ at $P$. Join $CD$, cutting circle at $D$ and $E$.

## No. 47

1. Triangle $ABC$, $PQR$ congruent; triangles $ADC$, $PSR$ congruent.
2. $EDB = 90° - EBD = 90° - ACB = BDF$.
4. $AD^2 = AB^2 + BD^2 = AC^2 + 2AC . CB + 2BC^2$, etc.
5. Bisect line joining centres at $C$; join $AC$, draw $PAQ$ at right angles to $AC$.

6 Draw isosceles triangle with angles at base double angle at vertex describe a circle about it Bisect angles at base

## No. 48

1 Triangles $BAQ$, $CAP$ are congruent

2 Let $QA$ be perpendicular to $YX$, $QB$ to $XZ$ Through $Y$ draw parallel to $XZ$, meeting $QB$ produced at $C$. $QC$ is perpendicular to $YC$, angle $CYZ = YZX = ZYX$ . $QC = QA$ $QA - QC = BC =$ constant.

3 Construct triangle $ABG$, having $AQ = \frac{2}{3} \times 4\ 6$, $BG = \frac{2}{3} \times 3\ 5$ $G$ is the intersector of medians $AD$ and $BE$, etc

4 (i) Produce $DC$ to $H$, making $CH = CD$. $XH$ cuts $AC$ at point $P$

(ii) Produce $FD$ to $K$, making $DK = FD$, let fall $KL$ perpendicular to $AC$ produced, and produce $KL$ to $M$, making $LM = LK$. $XK$ cuts $AC$ at point $Q$ After striking $AC$ the ball moves towards $K$, after striking $CD$ it moves towards $F$

5 Semi perimeter $12\frac{1}{2}$ in Lengths are $12\frac{1}{2} - 9$, $12\frac{1}{2} - 10$ $12\frac{1}{2} - 6$ $AP = 12\frac{1}{2}$ (see Paper 23 Question 6)

6 Take $P$ in arc $AB$ Join $HL$, $KL$, required to prove $PLK + PLH = 2$ right angles

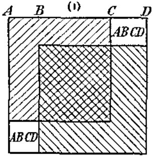
$PLK = PAK(PKAL$ cyclic$) = PBC(PABC$ cyclic$) = 180 - PLH(PLHB$ cyclic)

## No. 49

1 34 7 ft

2 $F$ is intersection of median of triangle $ABD$ $AF = \frac{1}{3} AC$, etc

3 (i)




(ii) $(a + b + c)^2 + b^2 = (a + b)^2 + 2ac + 2bc + c^2 + b^2$
$= (a + b)^2 + (b + c)^2 + 2ac$

4 Let $ABCD$ be quadrilateral and $H$ $K$ the mid points of diagonals $AC$ $BD$
$AD^2 + DC^2 = 2DH^2 + 2HC^2$ $AB^2 + BC^2 = 2CH^2 + 2BH^2$
$AB^2 + BC^2 + CD^2 + DA^2 = 4HC^2 + 4HK^2 + 4DK^2$
$= AC^2 + BD^2 + 4HK^2$

5 Radii to $B$ and $C$ make equal angles with $AC$ and are parallel tangents are parallel

Angle $CED$ = angle between tangent at $C$ and chord $CD$ = angle $CDE$ (*parallel*) $CE$ = arc $CD$

6 Draw circle, centre $I$ radius 1 3 in Draw any radius $IE$ and tangent at $E$ With centre $I$, radius 2 5 in., strike arc cutting tangent at $A$, and draw $AQ$ the other tangent Produce $AE$ to P, making $EP = 4$, draw $PH$ perpendicular to $AP$, meeting $AI$ produced at $H$ With centre $H$, radius $HP$, describe circle. Draw transverse common tangent meeting $AQ$ at $B$, and $AP$ at $C$.

## No. 50

1. $AP + PB > AB$, etc. $\therefore AP + BP + CP >$ half sum of sides.
$AP + PB < AC + CB$ (Paper 43, Question 1), etc.
$\therefore AP + BP + CP <$ sum of sides.

2. Let bisectors of $C$ and $D$ of quadrilateral $ABCD$ meet at $E$, and of $A$ and $B$ meet at $F$.
Angle $E = 180 - \frac{1}{2}C - \frac{1}{2}D$, angle $F = 180 - \frac{1}{2}A - \frac{1}{2}B$. $\therefore E + F = 180°$.
If $E = 90°$, $\frac{1}{2}C + \frac{1}{2}D = 90$, $C + D = 180°$, and $DA$ and $CB$ are parallel.

3. Draw $AB = 3{\cdot}4$, produce to $E$, making $AE = 6$. With centre $B$, radius 4, and with centre $E$, radius $4{\cdot}3$, describe arcs cutting at $C$. Complete parallelogram $AECD$. Area $= 18{\cdot}5$ sq. cm.

4. Circle on $OP$ as diameter. Longest chord is diameter, shortest is perpendicular to $OP$ and is of length 8 cm.

5. Draw any circle passing through given points $A$ and $B$, and cutting given circle at $P$ and $Q$. Produce $AB$, $PQ$, to meet at $R$. From $R$ draw tangent $RT$ to first circle. Circle $ABT$ is required circle.

6. If inscribed circle touches sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively, then circles with respective radii $AF$, $BD$, $CE$ are required circles.
$A = 120°$, $B = 20°$, $C = 40°$.

## No. 51

1. Triangles $ACP$, $BCQ$ are congruent. $\therefore AP = BQ$, etc.

2. Similar to Paper 49, Question 4.

3. Angle $AEC = ABC + BAD = \frac{1}{2}AOC + \frac{1}{2}BOD = \frac{1}{2}AOB + \frac{1}{2}BOD = \frac{1}{2}AOD$.

4. Draw isosceles triangle with base 12, height 8. Inscribe circle in it. Radius is 3 in.

5. $AR = \frac{2}{3}AQ$ and $AQ = \frac{2}{3}AC$. $\therefore AR = \frac{4}{9}AC$. $\therefore AR : RC = 4 : 5$.

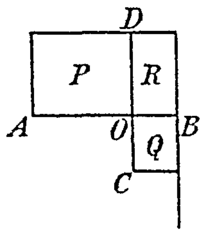
6. If $AB = 2$, $BC = \sqrt{3}$, $AC = 1$, $CD = 2 + \sqrt{3}$, $AD = \sqrt{8 + 4\sqrt{3}}$
$$= \sqrt{2}(\sqrt{3} + 1)$$
$$\therefore \sin 15° = \sin ADC = \frac{1}{\sqrt{2}\,(\sqrt{3} + 1)} = {\cdot}2588. \quad \tan 15° = \frac{1}{2 + \sqrt{3}}$$
$$= 2 - \sqrt{3} = {\cdot}2679$$

## No. 52

1. $PQ$ and $RS$ each parallel to one diagonal, so are $PS$ and $QR$. $\therefore PQRS$ is a parallelogram.

2. Right bisectors of $AC$ and $BC$ meet at $V$.

3. Angles $ADC$ and $ADB$ are each right angles.




5. Let $OA : OB = OC : OD$; place them as in the figure.
Then $P : R = OA : OB$
$Q : R = OC : OD$. $\therefore P = Q$.
Triangles $POA$, $PAC$ are similar.
$\therefore PC : PA = PA : PO$.

6. $PN = a - r\cos x°$, $QN = r\sin x°$, $PQ$
$$= \sqrt{a^2 + r^2 - 2ar\cos x°}$$
$\therefore PQ$ is greatest when $\cos x°$ is least, i.e. $x = 180$, and least when $x = 0°$.

## No 53

1 Let $PQ$ produced meet $AB$ at $C$ Triangles $APQ$ $BPQ$ are congruent (3 s des) angle $APC$ = angle $BPC$ triangle $APC$ $BPC$ are congruent
2 From triangles $DAB$ $CBA$ $BD > AC$ From triangles $DCB$ $ADC$ angle $BCD >$ angle $ADC$
3 $AC = 6\ 5$ angle $ADC$ is a right angle
Area $= \frac{1}{2} \times 5\ 6 \times 3\ 3 + \frac{1}{2} \times 6 \times 2\ 5 = 16\ 74$
4 A circle can be inscribed in the quadrilateral bisectors of angles meet at centre of circle
5 $2PQ = AB$ $2PR = 1C$ $PQ$ $PR = AB$ $AC = QX$ $XR$ etc
6 $PQ = PT \cos P$ (since $TQR$ is a right angle) $PT = TR \cot P$ (since $PTR$ is a r ght angle)
$PQ = 9\ 6 \cot 63°\ 15 \cos 63°\ 15 = 2\ 18$

## No 54

1 Four angles are needed.
2 Make $BC = 3\ 4$ angle $CBX = 4°°$ and cut off $BX = 1\ 6$ Draw perpendicular bisector of $CX$ meeting $BX$ produced at $A$ $AC = 3\ 3$
3 Draw $AB = 3\ 2$ produce to $C$ making $BC = 1\ 2$ At $B$ erect perpendicular meeting semi circle on $AC$ as diameter at $E$ From $BA$ cut off $BF - 1\ 7$ join $FE$ and draw $EG$ perpendicular to $EF$ meeting $AC$ produced at $G$ $BG$ is other s de of rectangle
Or find the fourth proport onal to 1 7 3 2 1 2
4 (i) Perpendicular bisector of $AB$
(ii) Circle with centre $C$ and radius — s de of square equal to $CA$ $CB$
5 Draw isosceles triangle s des 3 in 3 in 2 in and make a square equal to it suppose side is $x$ in Make another triangle similar to former with sides n ratio $x$ 2 5

## No 55

1 117° 108° 117° 99° 99°
2 Angle $DEA = EAB$ (parallel) $- DAE$ $DE - DA < AB < DC$
4 $BI$ produced passes through $D$ See Paper 35 Question 6
5 $BICE$ is cycl c since angles $IBE$ $CBE$ are right angles
triangles $BID$ $ECD$ are equiangular $BI$ $CE = DB$ $DE$
6 Height 14 ft 4 in. Inclination, 15° 29

## No 56

1 Angle $XOY$ is a right angle
2 Draw $CZ$ parallel to $XO$ meeting $OY$ at $D$ Along $DY$ mark off $DB = OD$ Join $BC$ and produce to meet $OX$ at $A$
3 Since rectangles are equal $A$ $B$ $C$ $D$ lie on a circle etc
4 See Paper 3 Quest on 5
$73^2$ $55^2 - 128 \times 18 = 64 \times 36 - 48^2$ triangle is right angled

$$\text{Area} = 48 \times 55 - 2 \quad \text{Hence radius} = \frac{48 \times 55}{176} = 15$$

5 (i) See Paper 52 Question 5
(ii) $PC$ $CO = CA^2 - CA$ $CB$ $XC$ $CY$
Hence $PC$ $CY = CX$ $CO$ and included angles equal triangles $PCX$ $OCY$ are similar
(iii) $PC$ $CO = XC$ $CY$ $P$ $X$ $O$ $Y$ are on a circle
(iv) Angles $OPX$ $OPY$ subtend equal chords $OX$ and $OY$
6 Distance from $A$ 9 03 miles distance from road 4 70 miles. Direction 58° 38 W of $S$

## No. 57

2. Draw through $A, B, C, D$ lines parallel to the diagonals.
3. (i) Each parallelogram $= \frac{1}{2}$ whole parallelogram − halves of two small parallelograms.
   (ii) Triangles $ABC$, $AYP$, $PKC$ are equiangular and similar.
5. See Paper 5, Question 5. Angle $P = \frac{1}{2}A + \frac{1}{2}B$, etc.
6. First make triangle $BCD$; on $BD$ draw segment to contain 106°, and in it place $BA = 3$ in. Area 9·8 sq. in. Sides are $\frac{3}{5}$ the sides of $ABCD$.

## No. 58

1. See Paper 20, Question 2.
2. See Paper 19, Question 4.
3. Draw $XY$ parallel to $AB$ at distance 2 in. On $AB$ as diameter describe semi-circle, in it place chord $BP = 1{\cdot}25$. Produce $AP$ to meet $XY$ at $D$, etc.
4. At $Q$ and $R$ draw perpendicular to the tangents to meet at $O$. Prove $OQ = OR$.
5. Divide $AB$ internally at $X$, externally at $Y$, in the ratio 2 : 1. Circle on $XY$ as diameter is required locus.
6. 69° 11′.

## No. 59

1. If $O$ is intersector of diagonals, triangles $BOC$, $DOC$ are equal, also triangles $BOE$, $COE$.
2. Sides of isosceles triangle make equal angles with line through vertex parallel to base; therefore their sum is a minimum. (See Paper 20, Question 2.)
3. (i) Bisect line.
   (ii) All the rectangles have same perimeter.
4. (i) Rectangles have same area.
   (ii) Perimeter least when chord is least, i.e. when chord is perpendicular to line joining point to centre.
5. At $A$ make $BAC = 22\frac{1}{2}°$, $AC$ meeting perpendicular at $B$ at $C$. Right bisector of $AC$ meets $AB$ at required point $P$.
6. Loci of vertices is circle on hypotenuse as diameter. ∴ area greatest when height = radius of that circle. Hence maximum area is $\frac{1}{4}c^2$.

## No. 60

1. $52\frac{1}{2}° = \frac{1}{2}(60° + 45°)$.
2. Join $CE$. Angles $KAE + BAC = 180°$.
   ∴ angles $KEA + ACB = 90°$. ∴ $KEC + BCE = 180$. ∴ $KE$ and $BC$ are parallel.
3. At $B$ erect perpendicular $BC = 2$ in. Draw $CD$ perpendicular to $AC$, meeting $AB$ at $D$. Mid-point of $AD$ is required point $P$.
4. Locus is circle having as diameter the radius to $A$.
5. Let common tangent at $C$, the point of contact, meet $AB$ at $D$. Then $AD = DC = DB$.
   Triangles $AHD$, $BDK$ are equiangular ($H$ and $K$ being the centres).
   ∴ $AH : BD = AD : BK$.
   ∴ $AD$ is mean proportional between radii and $AB$ between diameters.
6. Part is 14 in. long, i.e. $(7 + 12) - (12 - 7)$.
   Maximum value of $OAP = \tan^{-1}\frac{7}{12} = 30° 15′$.

## No. 61

1 *Draw* $BH$ *parallel to* $AD$ *meeting* $DC$ at $H$, bisect $BH$ at $K$ Join $EK$, $FK$
Then $EK$ and $FK$ are both parallel to $DC$
Also $EK = \frac{1}{2}(AB + DH)$ and $FK = \frac{1}{2}(HC)$

2 Construct triangle $CGH$ having $CG = \frac{4}{5} \times 1\ 9$, $GH = \frac{4}{5} \times 2\ 7$,
$$HC = \tfrac{4}{5} \times 3\ 4$$
Bisect $GH$ at $D$, produce $CD$ to $B$ making $DB = CB$, etc

3 Angles $ABC$, $ABD$, subtended by equal chords, are equal

4 *See Paper 23, Question 2* Inscribe circle in $ABC$, see Paper 50, Question 6

5 Angle $PDA = DBA$ (alternate segment) $= ADC$ (similar triangles since $ADB$ is a right angle)

$PA \quad AC = PD \quad DC = PB \quad BC$ (since $DB$, perpendicular to $DA$, bisects exterior angle)

6 $13^2 - 3\ 2^2 = 16\ 2 \times 9\ 8 = 3\ 24 \times 49 = 1\ 8^2 \times 7^2$ $\qquad BD = 12\ 6$
$4^2 - 3\ 2^2 = \ 7\ 2 \times \ 8 = 1\ 44 \times \ 4 = 1\ 2^2 \times 2^2$ $\qquad CD = \ 2\ 4$
$$BC = 15.$$
$ABC = 14°\ 15$, $ACB = 53°\ 8$

## No. 62

1 Cut off from $DB$, $DE = AB$ Draw $EP$ parallel to $DA$

2 Draw quadrilateral $ABCD$ Draw $CE$, parallel to $DB$, meeting $AB$ produced at $E$

Bisect $AD$ at $F$, draw $EG$, parallel to $BF$, meeting $AD$ at $G$ $BG$ bisects quadrilateral $ABCD$

3 Let circles with centres $A$, $B$, $C$, $D$ touch at $P$, $Q$, $R$, $S$ so that $APB$, $BQC$, $CRD$ $DSA$ are straight lines
Angle $APS = 90 - \frac{1}{2}A$, $BPQ = 90 - \frac{1}{2}B$ $\quad$ angle $SPQ = \frac{1}{2}A + \frac{1}{2}B$
Similarly, angle $SPQ = \frac{1}{2}C + \frac{1}{2}D$ $\quad SPQ + SRQ = \frac{1}{2}(A + B + C + D)$ $= 2$ right angles

4 Quadrilateral $AFDC$ is cyclic $\quad AK \quad KD = CK \quad KF$ etc

5 *See Paper 58 Question 5*

6 Along $AX$ cut off $AP$ such that $AP^2 = AB \quad AC = 75 \times 91$ Angle $BPC$ is maximum

Circle $BPC$ touches $AX$ at $P$ Let $O$ be its centre, and $OD$ the bisector of $BC$

$$\text{Angle } CPB = \text{angle } BOD\text{, } \tan BOD = \frac{8}{\sqrt{75 \times 91}} \qquad CPB = 5°\ 32$$

## No. 63

1 Take any point $P$ above $AB$ and between $A$ and $B$ With centre $P$ and radius $PB$ describe circle Produce $BP$ to meet circle at $C$ $AC$ is perpendicular to $AB$

2 Draw a transversal cutting $AX$ and $AY$ Bisect the four interior angles so formed Line joining the points of intersection of bisectors of angles is the required bisector

3 Bisect $AB$ at $C$ Draw $AQ$ parallel to $YB$ and $BQ$ parallel to $XP$ $QC$ produced is the diagonal and that would pass through $P$

4 In $AX$ take any point $P$ and mark off $PQ = 2$ in draw $PS$ parallel to $BY = 3$ cm, and complete parallelogram $PQPS$ Produce $QS$ to meet $BY$ at $T$ Bisect $QT$ at $K$, through $K$ draw a line parallel to $PR$, this is the required line (Use fact that median of triangle bisects all lines parallel to base)

5. Join $A$ and $B$ to any point $C$ on the arc $AB$. At $A$ make angle equal to $ABC$ and at $B$ make angle equal to $BAC$.

Join $P$ to $A$, cutting arc at $D$. Find $PE$ the mean proportional to $PA$ and $PB$. Circle with centre $P$, and radius $PE$, cuts arc at points of contact of tangents from $P$.

6. Distance $= 2\sqrt{5} = 4{\cdot}47$ in.

## No. 64

2. Draw $XY$ parallel to $AB$ at distance 2 cm. With centres $A$ and $B$, and radii each 3·5, draw circles meeting $XY$ at $D$ and $E$. $DE$ is the complete locus.

3. See Paper 7, Question 4.

4. Describe circle to touch three sides, and then *reductio ad absurdum*.

5. Find a line, by geometrical construction, $= \sqrt{5}$. Increase sides in ratio $\sqrt{5} : 1$.

6. Construct triangle $VAB$ with $VC$ perpendicular to $AB$. From $VA$ cut off $VP = 2$ cm., make angles $PVQ = 60°$, and $VPQ = 30°$. Produce $VQ$ to $R$, making $VR = QP$. Join $RA$ and draw $QS$ parallel to $RA$, meeting $VA$ at $S$. Draw $ST$ perpendicular to $VC$. Section is a circle with centre $T$ and radius $TS$.

Angle $AVB = 39° 14'$.

## No. 65

1. $P$ must be outside circle with centre $A$ and radius 1·7 in.; inside a circle with centre $B$ and radius 1·1 in.

2. Triangle $ABC$ is half each of the three parallelograms formed.

$XA$, a median of $XYZ$, bisects $BC$, the other diagonal of parallelogram $ABXC$, and is a median of $ABC$.

4. Draw intersecting circles, one passing through $A$ and $C$, the other through $B$ and $D$. The chord of intersection cuts $AD$ at $O$.

5. Bisect $XZY$ by line $ZO$ and make $ZXO$ equal to $CAP$.

6. Height $= 20 \times \sin 56° 41' \div \sin 5° 9' = 186{\cdot}3$ ft.

## No. 66

1. Triangles $ABC$, $A'BC'$ are equal in all respects.

Let $AC$, $A'C'$ meet at $D$, either or both being produced if necessary; and let $BA'$, $AC$ meet at $E$.

Angle $BAC = BA'C'$, $BEA = DEA'$. $\therefore ABA' = ADA'$.

2. Suppose $AB > AC$, then $Y$ is in $AC$ produced. Draw $CZ$ parallel to $AB$, meeting $XY$ at $Z$.

Then $CZY = AXD = CYZ$. $\therefore CZ = CY$. Triangles $XBD$, $ZCD$ are congruent. $\therefore CZ = BX$, etc.

Note that area of triangle $ABC$ is less than area of triangle $PAQ$ where $PQ$ is any line through $D$. The second part is solved by drawing $BC$ so as to be bisected at $D$. See Paper 6, Question 2.

3. Locus is common chord produced.

5. Triangles $BAD$, $PQS$ are similar (2 sides and incl. angle).

6. Suppose $P$ is between $A$ and $C$, consequently $Q$ between $E$ and $B$; then $R$ is between $D$ and $C$, and $S$ is in $DE$ produced. Prove $SQRP$ is cyclic.

$$\text{Angle } SQD = SQB + BQD = BAE + BCP$$
$$= RPC + RCP = DRP, \text{ etc.}$$

## No. 67

2 Draw $AB = 3\ 8$, on $AB$ as diameter describe a circle With centre $B$, radius 1 cut off $BC$ from $BA$ and $BD$ on $AB$ produced On $CD$ draw equilateral triangle $CDE$ Through $E$ draw $EF$, meeting circle on $AB$ at $F$ Let fall $FG$ perpendicular to $AB$ Then $AG\ \ GB = 3$ sq in

3 See Paper 36 Question 5

4 See Paper 34, Question 6

5 (i) $\frac{AP}{PB} = \frac{AQ}{QB} \quad \frac{AO+OP}{OB-OP} = \frac{AO+OQ}{OQ-OB} \quad \frac{AO}{OP} = \frac{OQ}{AO}$

*(componendo et dividendo)*

(ii) $\frac{PB}{AP} = \frac{QB}{AQ} \quad \frac{AB-AP}{AP} = \frac{AQ-AB}{AQ} \quad \frac{AB}{AP} + \frac{AB}{AQ} - 2$

## No. 68

1 16 right angles.

2 Let quadrilateral $ABCD$ be bisected by both $AC$ and $BD$
Then triangle $ADB$ = triangle $ACB$ $AB$ and $CD$ are parallel, etc

3 The lines are the parallels to $BC$ at distance equal to radius of given circle

4 See Paper 60, Question 3

5 Suppose $P$ is mid point of side $AB$ of a rectangle $ABCD$ and $Q$ of side $AD$ Bisect $PQ$ at $R$ Locus of $A$ is circle centre $R$ radius $\frac{1}{2}$ in., Locus of $C$ is a circle, centre $R$ radius $1\frac{1}{2}$ in

Produce $PQ$ both ways to $S$ and $T$ so that $PS - PQ - QT$

Locus of $B$ is circle on $PS$ as diameter and of $D$ circle on $QT$ as diameter

6 From any point $X$ on $BA$ let fall $XY$ perpendicular to $BC$ and complete square $XYZW$

Join $BW$ and produce to meet $AC$ at $Q$ Let fall $QR$ perpendicular to $BC$, etc

From triangles $APQ$ $ABC$, $\frac{x}{BC} = \frac{AP}{AB} = \frac{SD}{BD}$

From triangles $PSB$, $ADB$ $\frac{x}{h} = \frac{BS}{BD} \qquad \frac{x}{BC} + \frac{x}{h} = 1$

## No. 69

2 Perpendicular from $C$ on $BD$ is twice $AB$ $C$ is outside square beyond $BD$ or beyond $AE$ angle $BAC$ or $ABC$ is obtuse

If $BAC$ is obtuse $AE$ bisects $BC$ if $ABC$ is obtuse, $BD$ cuts $AC$ in a point of trisection

3 Length $= \sqrt{15} = 3\ 87$ in

5. Angle $PYB = 180° - PAB = QAB = 180° - QXB$ etc

6 Radius of circle is constant in length locus is a circle centre $O$

## No 70

1 See Paper 2 Question 1

Or produce $AD$ to $E$, making $DE - AD$ Join $EC$

Triangles $ADB$ $CDE$ are congruent $AB = CE$, also angle $CAD = BAD = DEC$ $CE = AC$

2. Produce $BA$ to $E$, making $AE = AC$. Then $AD$ bisects $EC$ at right angle. $\therefore DE = DC$, etc.
3. Draw $CD = 2{\cdot}5$. On $CD$ as diameter describe semi-circle, and in it place chord $DE = 1$. Produce $DE$ to $B$, making $DB = 3$; draw $CA$ parallel to $EB$ and equal to 2.
$AB^2 = CE^2 = 2{\cdot}5^2 - 1^2 = 3{\cdot}5 \times 1{\cdot}5$. $AB = 2{\cdot}29$ in. Area $= 5{\cdot}72$ sq. in.
4. $7{\cdot}4^2 - 7{\cdot}0 = 14{\cdot}4 \times {\cdot}4 = 2{\cdot}4^2$, etc.
5. See Paper 41, Question 6.
6. $OA = 41{\cdot}32$ ft., $OB = 37{\cdot}32$ ft., $OC = 41{\cdot}73$ ft., $OD = 37{\cdot}73$ ft.

## No. 71

2. See Paper 33, Question 2.
4. If $BC = CA$, angle $CAB$ = angle $CBA$ = angle $DBC$. $\therefore DB$ is a tangent. If $P$ between $B$ and $D$, then angle $DBC <$ angle between $CB$ and tangent at $B$. $\therefore$ angle $CBA < CAB$, $\therefore AC < CB$.
5. Triangle $ACD$ : triangle $ABD = 6^2 : 4{\cdot}5^2 = 16 : 9$ (similar triangles).
6. Height $= 134{\cdot}6$ ft. Angle of elevation $= 12° \; 48'$.

## No. 72

1. Angles $A + B + C + D = 360$.
Angle $X = 180 - A - D$. Angle $Y = 180 - C - D$. $\therefore X + Y = B - D$.
Let $OX$ cut $BC$ at $E$.

$$\text{Angle } XOY = 180 - \frac{Y}{2} - OEB = 180 - \frac{Y}{2} - \frac{X}{2} - EBX = B - \frac{B-D}{2}$$
$$= \frac{B+D}{2}.$$

3. $15^2 - 13^2 > 7^2$. $\therefore$ triangle is obtuse angled. Perpendicular $= 6{\cdot}062$.
4. Describe circle, centre $O$, radius $3\frac{1}{2}$ in.; describe circle, centre $P$, radius 2 in. The two points of intersection of these circles are the possible centres. Distance $= 3{\cdot}92$ in.
6. Angle $= 2\cos^{-1}\dfrac{7{\cdot}2}{8} = 2\cos^{-1}{\cdot}9 = 51° \; 41'$.

## No. 73

2. Quadrilateral $ABCD$ is such that triangles $ABC$, $ADC$ are equilateral.
Draw $BE$, parallel to $AC$, meeting $DC$ produced at $E$.
Mark off $DF = 2{\cdot}8$. Draw $EG$ parallel to $FA$ to meet $DA$ produced at $G$. Triangle $DFG$ is equal to $ABCD$.
3. (i) Circle having as diameter line joining centre of given circle to given point.
(ii) Diameter perpendicular to given line.
4. $AP$ bisects angle $BAC$. $\therefore PX = PY$. Also $PB = PC$. $\therefore BX = CY$.
5. Let $XAY$ be tangent at $A$ to circle $ABC$.
Angle $XAB = ACB = AQP$. $\therefore XAY$ touches circle $APQ$.
For second part, draw tangent at $O$ to circle $POQ$ and similar proof follows.
6. $XO$ bisects angle $AXY$, and $YO$ bisects supplementary angle $XYB$, $\therefore XOY$ is a right angle. $\therefore$ angle $BOY$ = angle $OXA$. $\therefore$ triangles $OXA$, $OYB$ are similar.
Hence $BY : OB = OA : AX$. $\therefore BY = 2OA = 2BQ$. $\therefore YQ$ = radius.

## No. 74

2 See Paper 62, end of Question 2

4 Centre is intersection of perpendicular bisectors of $AB$ and $CD$

5 $CB$ is tangent to circle $A$ angle $BCE$ = angle in alternate segment $= \frac{1}{2}$ angle $CAB$ Similarly, $ACD = \frac{1}{2}CBA$

. $ACD + BCE = \frac{1}{2}(CAB + CBA) = \frac{1}{2}$ a right angle $\therefore DCE = \frac{1}{2}$ a right angle

6 $R = \dfrac{a}{2 \sin A} = \dfrac{abc}{2bc \sin A} = \dfrac{abc}{4\triangle}$

For $r$, see Paper 3, Question 5

Let $O$ be circumcentre, $I$ the incentre of triangle $ABC$

Produce $OI$ both ways to cut circumcircle at $P$ and $Q$

Then $R^2 - OI^2 = PI \; IQ$

Produce $AI$ to meet circle at $D$ Join $DO$ and produce to meet circle at $E$ Join $DC$, $DC$ is perpendicular to $EC$

Let fall $IX$ perpendicular to $AC$

Triangles $CDE$, $AIX$ are similar (equiangular)

$DE \; AI = DC \; IX$, i.e. $2Rr = AI \; DC$

But $DC = DI$ (Paper 35, Question 6)

Hence $R^2 - OI^2 = PI \; IQ = AI \; DI = AI \; DC = 2Rr$

## No. 75

1 With radius 1 in, describe circles with centres $A$ and $B$ cutting at $P$ With radius 1 in, describe circle with centre $P$ cutting circle with centre $B$ at $Q$

Circle with centre $Q$ and radius 1 in, cuts circle with centre $P$ at point $C$

2 Draw $OPQ$, parallel to $DA$ and $CB$, cutting $CD$ at $P$ and $AB$ at $Q$

Triangles $DOA + AOB + BOC$ = triangle $DOC$ + parallelogram $ABCD$, but $DOA = \frac{1}{2}$ parallelogram $DQ$ and $BOC = \frac{1}{2}$ parallelogram $CQ$

Triangles $DOA$ and $BOC = \frac{1}{2}$ parallelogram $ABCD$ = triangle $DAC$

Triangle $AOB$ = triangle $DOC$ + triangle $DAC$

= triangle $AOC$ + triangle $AOD$

3 $DA$ is a median of triangle $BDE$, so is $EC$ $BF$ is the third median

4 (i) Make angle $= 360° - 15 = 24$ at centre, this gives the side of figure

(ii) Let $AB$ be side of inscribed equilateral triangle and $AC$ a side of inscribed regular pentagon Bisect arc $BC$ at $D$, then $BD$ $DC$ are two consecutive sides of the required figure

5 By similar triangles $OA : OD = OB \; OC = 3 \; 9$

Dividendo $OA \; AD = OB \; BC = 3 \; 6$ $OA = 2$ cm, $OB = 2\frac{1}{2}$ cm

6 Cos $BCD = \frac{3}{4}$ $BCD = 41° 24$

Area $= 6 \times 5 \sin BCD = 19\ 84$ sq cm

## No 76

3 $APQR$ is a parallelogram $AR$ bisects $PQ$

5 Let $P$, $Q$, $R$ be respective circumcentres of triangles $ABC$, $ABD$, $AC$ Then $PQ$ is perpendicular to $AB$ and $PR$ is perpendicular to $AC$ angle $QPR = 180 - BAC$

Angle $AQD$ at centre $2ABC$ $AQR = ABC$

Similarly, $ARQ = ACB$ $QAR = BAC$, etc

6 Find fourth proportion to 3 1 cm, 2 3 cm, 1 8 cm value = 1 34

## No. 77

2. See Paper 20, Question 2.
3. Draw any chord of length 5·2. Draw concentric circle touching this chord, and from $P$ draw tangent to this circle.
4. Draw any circle with centre $O$ passing through $AB$; at $A$ draw $AC$ a tangent of length 4 in. With centre $O$ and radius $OC$, describe circle cutting $AB$ produced in $P$.
Or bisect $AB$ at $D$, at $B$ erect perpendicular $BE = 4$. With centre $D$, radius $DE$, describe circle cutting $AB$ produced at $P$.
5. Triangle $ABC$, $XYZ$ are equiangular (angles in same segment).
6. Inscribed polygon is made up of $n$ isosceles triangles with equal sides, equal to $r$ and included angle $= \dfrac{360°}{n}$.
Described polygon is made up of $n$ isosceles triangle with height $r$ and angle between equal sides $\dfrac{360°}{n}$. Perimeter $= 2nr \tan \dfrac{180°}{n}$; area $= nr^2 \tan \dfrac{180°}{n}$.

## No. 78

1. Angle $ACQ = 180° - APQ = BPQ = 180° - BDQ$, etc.
2. In right-angled triangle $POQ$, $OR = RP = RQ$. $\therefore$ locus of $R$ is quadrant of a circle with centre $O$ and radius $\frac{1}{2}$ of $PQ$.
3. For isosceles triangle, produce $DC$ both ways, draw $BH$ parallel to $AC$ to meet $DC$ at $H$, and $EK$ parallel to $AD$ to meet $CD$ at $K$. $AHK$ is isosceles triangle of equal area.
4. Let circles $XBC$, $YCA$ meet at $O$.
Angle $BOC = 180° - X$; angle $COA = 180° - Y$. $\therefore$ angle $AOB = X + Y$.
$\therefore AOB + AZB = 180°$, and circle $ABZ$ passes through $O$.
6. If $AB^2 = AQ \cdot BR$, then $AQ : AB = AB : BR$. $\therefore$ try to prove triangles $ABQ$, $ABR$ to be similar.
By data, $RP \cdot RA = RB \cdot RC$. $\therefore ABCP$ is cyclic. Hence $PAQ = PBR$. $\therefore$ angle $BAP$ = angle $AQB$, etc.

## No. 79

1. Angles $ICE$, $IBE$ are right angles. $\therefore B, I, C, E$ lie on a circle.
Also $AIE$ lie on bisector of $BAC$, which bisects arc $BC$ of circumcircle at $D$. Now $DB = DC = DI$ (see Paper 35, Question 6). $\therefore D$ is the centre of circle $BICE$.
2. Make triangle $ABC$. Produce $BA$ to $P$ so that $BA = AP$, and $BC$ to $Q$ so that $BC = CQ$; then $PQ$ is parallel to $AC$. Make $BCD = 95$, $D$ being on $PQ$.
3. $AX \cdot AY = AX^2 + BX \cdot XY = BX^2 + BX^2 + BX \cdot BY$.
4. Within angle $XOY$ draw $CH$ parallel to $OX$ at distance equal to radius of given circle, and $CK$ parallel to $OY$ at distance equal to radius. These parallels are loci of centres of the circle. Distance between centres is bisected by $R$, the point of contact. $\therefore$ locus is quadrant of circle with centre $C$ and radius equal to radius of given circles.
5. Use angle properties of cyclic quadrilaterals.
6. Triangles $PXZ$, $PXY$ are equiangular.

## No. 80

1 $D = F = \frac{8}{7}$ of right angle, $C = G = \frac{6}{7}$ of right angle

2 Draw $BPQ$ parallel to $YZX$, cutting $CZ$ at $P$ and $AX$ at $Q$, then $AQ = 2CP$ $AX + BY = AQ + QX + BY = 2CP + 2PZ = 2CZ$
In second case, $AX - BY = 2CZ$
3 $DE^2 = AE^2 + AD^2 = BC^2 + AD^2 = AB^2 + AC^2 + AD^2$
Similarly for $CF^2$
4 Draw $OT$ common tangent at $O$
Angle $ROS = ROT - SOT = RQO - SPO$ (alternate segment) $= POQ$
5 $PC \; PD = PA \,.\, PB = PT^2$, etc
6 On $BC$ as diameter describe semi circle cutting $CA$ at $E$ In semi-circle place chord $CD = 2BE$

## No. 81

1 By congruent right angled triangles, triangles $ABC$, $PQR$ have equal altitudes etc
2 Let $ABCD$ be the parallelogram $E$ the intersection of the diagonals, $AP$, $BQ$, $CR$, $DS$, $EO$ the respective depths
By Question 2, Paper 80, $AP + CR = 2EO$ and $BQ + DS = 2EO$
3 $PMBK$ = triangle $ABC$ − triangle $AMP$ − triangle $PKC$
= triangle $ADC$ − triangle $AHP$ − triangle $PLC$ = $PLDM$.
Make rectangle $PLDH$ having $HP = 5$ $PL = 1$, produce $HP$ to $K$, making $PH = 2\ 3$ Complete figure with lettering of first part of this question, then $PMBK$ is required rectangle
4 In circle $A$ place chord = 5 cm and draw concentric circle touching chord
In circle $B$ place chord = 3 cm, and draw concentric circle touching chord.
*PQRS is a common tangent to these concentric circles*
5 See Paper 48, Question 6
6 Length $= \sqrt{20^2 + 15^2 + 12^2} = 27\ 7$ ft Angle $= \tan^{-1} 48 = 25° 38'$

## No. 82

1 Draw any line $XCY$ through $C$ From $A$ draw line $AX$, making $AXC = 60°$, from $B$ draw $BY$, making $BYC = 60°$ Produce $YB$, $XA$ to meet at $Z$
2 Let bisectors of angles $A$ and $B$ of parallelogram $ABCD$ meet at $P$ Then $A + B = 2$ right angles $APB$ is a right angle, etc
3 Draw $HK$ parallel to $PQ$ at distance 2, cutting $OS$ at $L$
Draw $XY$ parallel to $PQ$ at distance 2, cutting $OR$ at $Q$
Required locus is made up of the bisectors of the angles $HLS$, $KLS$, $XZR$, $YZR$
4 Divide $AB$ at $C$ so that rectangle $AB \; BC = AC^2$
5 $AK = \dfrac{AE}{\cos(90° - C)} = \dfrac{C \cos A}{\sin C} = \dfrac{a \cos A}{\sin A} = a \cot A$
Since $PQ$ is of constant length angle $PAQ$ is constant $AK$ is constant
6 $D$ is on perpendicular bisector of $BC$ Prove triangles $ABD$, $ACE$ to be equiangular

## No. 83

1. See Paper 33, Question 1.
2. Produce $BO$ to meet $AC$ at $D$; then $BOC > BDC > BAC$.
Angle may be outside triangle and yet $> BAC$.
4. Opposite angles equal and supplementary.
Mid-points are corners of a parallelogram, sides parallel to diagonals, which must be a rectangle. $\therefore$ diagonals are at right angles.
5. Triangles $ABC$, $BCD$ are equiangular.
6. Shadow $= 38{\cdot}60$ ft.
Sun is due south. Shadow of wall is parallelogram, length 90 ft., height 7 tan 40 sin 45. Area $= 374$ sq. ft.

## No. 84

2. Let fall $AE$, $BF$ perpendicular to $CD$ ($AB$ being less than $DC$). Right-angled triangles $ADE$, $BCF$ are congruent. $\therefore$ angle $ADE =$ angle $BCD$. Hence triangles $BCD$, $ADC$ are congruent. $\therefore$ angle $DAC =$ angle $DBC$, etc.
3. $\because$ $AB$ is divided at $P$, $AB^2 + BP^2 = AP^2 + 2AB \,.\, BP$
$$= 2BP^2 + 2AB \,.\, BP.$$
$\therefore$ $AP^2 = 2BP^2$ and construction is same as Paper 59, Question 5.
4. Draw $KXL$ not parallel to line of centres. Bisect $KX$ at $M$, $XL$ at $N$; then $MN$ is $\frac{1}{2}KL$ and is less than line joining centres. If $PXQ$ is parallel to line of centres, then $\frac{1}{2}PQ =$ line joining centres.
5. On side of $PQ$ remote from $A$ describe segment containing angles equal $90 + \frac{1}{2}PAQ$; draw bisector of $PAQ$ to meet arc of segment at $I$. $I$ is centre of inscribed circle, etc.
6. $AOXY$ is cyclic. $\therefore$ $OYX = OAX$.
$BOZX$ is cyclic. $\therefore$ $OZX = OBA$, but $OBA = OAX$, etc.

## No. 85

1 Draw triangle $ECB$ having $EC = 5{\cdot}6 - 3{\cdot}7$, $CB = 2{\cdot}5$, angle $CEB = 70°$.
Produce $CE$ to $D$, making $CD = 5{\cdot}6$. Complete parallelogram $DEBA$.
Area $= 10{\cdot}5$ sq. cm.
2. Their altitudes are equal, by congruent right-angled triangles.
4. See Paper 23, Question 6 (ii).
5. Suppose $D$ were outside circle $ABC$, then angles $ABC$, $ADC$ (opposite equal chords) would be equal, which is impossible.
6. $AI : ID = AB : BD = AC : CD = AB + AC : BC$.
Or produce $BA$ to $E$, making $AE = AC$. Then angle $BAC = 2$ angle $AEC$
$\therefore$ $AD$ is parallel to $EC$.
Hence $AI : ID = AB : BD = EB : BC = AB + AC : BC$.

## No. 86

1. Let fall $AD$ perpendicular to $XY$ and produce to $E$, making $DE = DA$
$EB$ produced cuts $XY$ at required point $C$.
Or, if $AB$ cuts $XY$ at $D$, draw locus of point $P$ such that $AP : PB = AD : DB$. Locus cuts $XY$ at $C$.
2. Each side of quadrilateral $= \frac{1}{2}$ diagonal of rectangle.
Each triangle cut off at corner $= \frac{1}{8}$ of rectangle.
3. Triangles $PAC$, $BAQ$ are congruent. $\therefore$ angle $RCA =$ angle $RQA$, etc.

4. *See Paper 19, Question 4.*
Projection of medians of triangles $ABC$, $ABD$, which bisect $AB$, are equal and on same side of mid point of $BC$

5 See Paper 3, Question 5

6 Triangles $PCA$, $OCA$ are equiangular
$PC \cdot CO = AC^2 = XC \cdot CY$   $P, X, O, Y$ lie on a circle
$\therefore XPO = XYO = OXY = OPY$.

## No. 87

1 *Triangles $DAE$, $CBE$ are congruent* $\therefore$ *triangles $DFE$, $CFE$ are congruent*

2 Draw through $B$ and $D$ parallels to $AC$, bisect $AC$ at $P$ and let perpendicular bisector cut parallel through $B$ at $Q$ and that through $D$ at $R$ Produce $PQ$ to $V$, making $QV = PR$   $VAB$ is required isosceles triangle

3 Radius of sphere = radius of circle inscribed in triangle of sides 13, 13, 10
Radius $= \frac{60}{18} = 3\frac{1}{3}$ in.

5 Bisect $PB$ at $X$, draw $XY$ parallel to $BC$, meeting $AC$ at $Y$. Join $PY$ and produce to meet $BC$ produced at $Q$

6 (i) Draw isosceles triangle $VAB$ having $VAB = VBA = 70°$   On $VC$, the perpendicular from $V$ to $AB$, describe a semi circle and in it put $CD = CB$   $VCD$ is required angle
(ii) Angle is $\cos^{-1}(\cot 70°) = 68° 39$

## No. 88

1 $AC = BC = 7\ 4$ *cm*

3 Triangle $DOC$ is $\frac{1}{2}$ an equilateral triangle   $OC = 2OD$   But $OA = OC$ ($O$ is on bisector of angle $B$, which is perpendicular bisector of $AC$)   $OA = 2OD = 2OE$, etc.

5 Triangle $ABE$, $CBD$ are congruent (2 sides and included angle)
$BAE = BCD$ $\therefore$ $AE$ produced passes through $D$

6 Height of triangle $ABC = 5 \tan 55°$   $AC = BC = 7\ 414$.

## No. 89

1 Triangles $ABC$, $DBC$ are congruent angle $DBC$ = angle $ACB$ etc.

2 Let $O$ be mid point of $BC$   Angle $BPC$ is a right angle   $BC = 2OP$ But $P$ is equidistant from $AB$, $BC$, and $CD$   $OP$ is parallel to $BA$
$BC = 12$ in

3 Side of square = 1 65 sq in

4 From $O$ the centre of circle, let fall a perpendicular to line through $Q$ perpendicular to $PQ$, perpendicular from $O$ cuts circle at $R$ and $S$   Join $RQ$, cutting circle at $T$, and $SQ$ cutting circle at $V$   $OT$ cuts $PQ$ at $X$ and $VO$ cuts $PQ$ at $Y$   Circles with centres $X$ and $Y$ satisfy conditions.

5 Triangles $BOC$, $EOC$ are congruent   $OBC = OEC$
Triangles $BOA$, $DOA$ are congruent   $OBA = ODC$, etc

6 Let $ABCD$ be cyclic quadrilateral   Make angle $DAE$ = angle $BAC$, $E$ being on $BD$
Triangles $DAE$, $CAB$ are similar   $AC \cdot DE = BC \cdot AD$
Triangles $BAE$, $ADC$ are similar   $AC \cdot BE = AB \cdot DC$, etc
In second part, $PABC$ is cyclic   $PA \cdot BC = AB \cdot PC + AC \cdot PB$, etc

## No. 90

1. $BC > BQ > PQ$.
3. Perpendicular from $P$ on $AB < AP <$ side of square.
Angle $APQ = 120°$; angle $APD = 75°$. $\therefore DPQ = 45°$.
4. Triangles $PAQ$, $HBK$ are congruent (chords $PQ$, $HK$ equidistant from centres).
$\therefore$ angle $APQ =$ angle $BHK$. $\therefore AP$ and $BH$ are parallel. $\therefore PH = AB$, etc. Or, suppose circles coincide; as $B$ circle is moved and centre travels from $A$ to $B$, $H$ travels from $P$ to $H$ and $K$ from $Q$ to $K$. $\therefore PH = AB = QK$.
5. $AE : EB = AD : DB = AD : DC = AF : FC$. $\therefore EF$ is parallel to $BC$.
6. Let fall $AZ$ perpendicular to $XY$, along $XY$ measure $ZB = 2{\cdot}5$ cm. Angle $ABX$ is 31°.

## No. 91

1. Triangles $BAD$, $CAD$ are congruent. $\therefore$ angle $BAC$ is bisected by $AD$ produced.
2. By proof of concurrency of the medians, lines respectively equal to $\frac{2}{3}$ of the medians form a triangle; hence any two medians are greater than the third.
Produce $AD$ to $E$, making $DE = AD$. Then $AC + CE > AE$; i.e. $AC + AB > 2AD$. $\therefore$ sum of medians less than perimeter of triangle.
3. Locus is line parallel to $XY$, 1·5 cm. from $XY$, between $O$ and $XY$.
4. Draw $AD$ perpendicular to $BC$, then $BD = DC$.
$\therefore AB^2 - AO^2 = BD^2 - OD^2 = BO \,.\, OC$.
5. Line bisecting angle between two lines bisects angle between perpendicular drawn to lines at angular point. Hence $PAQ$ bisects an angle between the radii drawn to $A$. Let $X$ be centre of $P$ circle and $Y$ of the $Q$ circle. Then triangles $PXA$, $QYA$ are equiangles. $\therefore PA : AQ = AX \; AY$.
6. (i) Triangle $AOB$ . triangle $AOC = AB : AC$.
$\therefore AO \,.\, OB \sin AOB : AO \,.\, OC \sin AOC = AB : AC$.
(ii) Triangle $AOB +$ triangle $BOC =$ triangle $AOC$, etc.

## No. 92

2. Along $BC$ measure $BD = 2$ in., draw $CE$, parallel to $DA$, cutting $BA$ at $E$. Construct triangle $PQR$ having $PQ = BD = 2$ in., $PR = BE$, $QR = DE$.
3. Draw $PE$ perpendicular to $BC$, and bisect $BC$ at $D$.
$\cdot 7 = PB^2 - PC^2 = BE^2 - EC^2 = (BE + EC)(BE - EC) = 2 \times 2{\cdot}1 \times DE$.
$\therefore DE = \frac{1}{6}$ of 1 in. $E$ is between $D$ and $C$ and perpendicular at $E$ to $BC$ is the locus of $P$.
4. Suppose circles touch externally. On side of given line opposite to given circle draw a line $XY$ parallel to given line at distance = radius of given circle. Then centre of any circle touching externally is equidistant from $XY$ and the centre of given circle. Similarly if circles touch internally, etc.
5. Join $B$ mid-point of line joining centres to $A$, required line is perpendicular to $AB$.
On $AC$ and $BC$ as diameters draw circles. Through $C$ draw $PCQ$, cutting circles at $P$ and $Q$ and bisected at $P$. Then $PCQ$ is middle line of square and square can be completed.
6. Triangles $PAB$, $PAC$ are between same parallels. $\therefore$ areas are as $PB : PC$. Triangles are equiangles. $\therefore$ areas are as $AB^2 \,.\, AC^2$.

## No 93

1 $AC = 926$ in , $BC = 1\ 426$ in

2 $EA = FG = HB$ . $EAHB$ is a parallelogram
Hence $FGHB = EAHB = 2$ triangle $AEB = ABCD$

3 Draw a line $OB$ and in it take any points $A$ and $C$, $C$ being between $A$ and $B$ Let $OA = a$, $OB = b$, $OC = c$ We have to prove that

$OA\ CB + OB\ AC = OC\ AB.$

*By first part,* $OA\ CB = OA\ OB - OA\ OC$
and $OB . AC = OB . OC - OB\ OA$
and $OC\ AB - OC\ OB - OC\ OA$

4 $DE$ is bisected at right angles by $AC$ and $DF$ is bisected at right angles by $AB$ See Paper 30, Question 6

5 Let ratio be $l\ m$, and $A$, $B$ the two given points Divide $AB$ internally at $P$ so that $AP\ PB = l\ m$ and externally at $Q$ so that $AQ\ QB = l\ m$ Circle on $PQ$ as diameter is required locus

6 Triangles $BRP$ and $PQC$ are equiangular and therefore similar
$SP\ SB = PQ\ BR = QC\ PR = SC\ SP$

## No. 94

1 If $XOY$ is obtuse, angles $YOA$, $XOB$ are equal bisectors do bisect angles between perpendiculars If $XOY$ is acute, produce $XO$ to $Z$, then $YOA = ZOB$, etc

2 Triangles $POA$, $QOC$ are congruent
Produce $AC$ to $D$ so that $CD = CA$, and $BC$ to $E$ so that $CE = CB$ Then $D$ and $E$ are on the other sides of rhombus
On $AB$ describe a segment on side remote from $C$, containing an angle 60°, complete circle and bisect minor arc at $F$ and produce $CF$ to cut major arc at $P$ $P$ is one vertex of rhombus Produce $PC$ to $R$ so that $CR = CP$ $R$ is the opposite vertex Produce $PA$, $RE$ to meet at $Q$ and $PB$, $RD$ to meet at $S$ $PQRS$ is the rhombus Side of rhombus is 3 3 in

3 Let $AB$ be a given line, at $B$ draw $BC$ perpendicular to $AB$, equal to $AB$ With centre $O$, the mid point of $AB$, radius $OC$, draw circle cutting $AB$ produced at $P$

4 If $A$ is centre of $PQX$ and $B$ of $PQY$ angles $APX$, $BPY$ are equal, since each equals $APY - 90°$, hence angles $PAX$ $QBY$ are equal
Also $PQY = \frac{1}{2}PBY$ and $PQX = \frac{1}{2}(360 - PAX)$ $PQY + PQX = 180°$

5 Triangles $PAX$, $PBY$ are equiangular

6 Height $= 136\ 3$ ft

## No. 95

1 Let $AG$ perpendicular to $DE$, cut $BF$ at $H$
Triangles $ADE$, $ABF$ are congruent
angle $AFB = AED = HAF$ $HF = HA$
Also angle $HAB = ABH$ $HB = HA$ $FB$ is bisected at $H$

2 Angle $PBQ = \frac{1}{2}$ right angle $= PQB$ $PQ = PB$
. $AP^2 + PB^2 = AP^2 + PQ^2 = AQ^2$
$AP^2 + PB^2$ is least when $AQ$ is least, i e when $P$ is at $O$ the mid point of $AB$.

3 Also $AP^2 + PB^2 = AQ^2 = AC^2 + CQ^2 = 2AO^2 + 2OP^2$

4 See Euclid III, Prop 21

5. See Paper 52, Question 5.
Produce $AD$ the perpendicular from vertex of an isosceles triangle $ABC$ to meet circumcircle at $E$. Triangles $ABD$ and $ABE$ are similar and

$$AE \cdot AD = AB^2.$$

6. Distance from vertex $= \frac{1}{2} \tan 72 = 1{\cdot}5389$.
Angle of pentagon $= 108°$, and of hexagon.
Let $ABCDE$ be pentagon and $CDFGHK$ be hexagon.
Then $CB$ and $DE$ fall inside hexagon.
Distance from $CD$ of $A$ is $1{\cdot}5389 \times CD$. Distance of $HC$ from $CD$ is $1{\cdot}732 \times CD$. $\therefore$ $A$ falls inside hexagon.

## No. 96

1. Proof is same as that for proving that exterior angle of a triangle is greater than an interior opposite angle.
2. Draw $OZ$ equal and parallel to $AB$. Draw $ZC$ parallel to $OX$, meeting $OY$ at $C$. Draw $CB$ parallel to $ZO$, meeting $OX$ at $D$. $ABCD$ is required parallelogram.
3. Equate two expressions for $OA^2 + OB^2 + OC^2$.
$AZ^2 + BX^2 + CY^2 = AO^2 + BO^2 + CO^2 - OZ^2 - OX^2 - OY^2$.
Let $P$, $Q$, $R$ be mid-points of $BC$, $CA$, $AB$ respectively.
Then $AO^2 + BO^2 = 2AR^2 + 2RO^2$.
Hence

$$\begin{aligned} AZ^2 + ZB^2 + BX^2 + XC^2 + CY^2 + YA^2 &= 2(AO^2 + BO^2 + CO^2) - 2(OX^2 + OY^2 + OZ^2) \\ &= 2(AR^2 + BP^2 + CQ^2) + 2(RO^2 - OZ^2 + OP^2 - OX^2 + OQ^2 - OY^2 \\ &= \text{constant} + 2(RZ^2 + PX^2 + QY^2). \end{aligned}$$

$\therefore$ L.H.S. is a minimum when $O$ is centre of circumcircle.
5. $BP : PC =$ triangle $AOB$ : triangle $AOC$, etc.
6. (i) Triangles $PRQ$, $ADC$ are similar.

$$\therefore \frac{AB}{RQ} = \frac{AD}{PQ} = \frac{AC}{PR}.$$

Also $RQ \cdot AP + PQ \cdot AR = PR \cdot AQ$. (Ptolemy's Theorem, Paper 89, Question 6.)
$\therefore AB \cdot AP + AD \cdot AR = AC \cdot AQ$.
(ii) $AD^2 + AB^2 = AC^2$. $\therefore AD \cdot AR + AD \cdot DR + AB \cdot AP + AB \cdot BP = AC \cdot AQ + AC \cdot CQ$.
$\therefore AB \cdot BP + AD \cdot DR = AC \cdot CQ$.

## No. 97

2. 2·8 in.
3. See Paper 20, Question 2.
Let $XYZ$ be any points in $BC$, $CA$, $AB$ respectively.
If $XZ$, $YZ$ are not equally inclined to $YZ$, find $O$ so that $OX$, $OY$ are equally inclined to $AB$. Then $OX + XY + YO < ZX + XY + YZ$. Hence if the sides of $XYZ$ are not equally inclined at $X$, $Y$, $Z$ to the sides $BC$, $CA$, $AB$ respectively, a triangle of smaller perimeter can always be found. Hence the triangle formed by joining the feet of perpendicular has the minimum perimeter.
4. Bisector of angle and perpendicular bisector of side both bisect same arc. $AD$ is common chord of the two circles.
5. See Paper 39, Question 4.
6. (i) Square = rectangle 5 in. by 3 in.; call side $x$. (ii) Call side of square $y$. Draw new equilateral triangle by increasing side 1 in. in ratio $x : y$. Side of required triangle is = 5·9 in.

## No. 98

1 *Draw isosceles triangle $ABC$ having $AB = AC = 3$ in., $BC = 2\ 2$ in* Draw $AH$ perpendicular to $BC$ and produce $AH$ to $K$, making $AK = 5$ in, etc

2 If $P$ is between $B$ and $D$, the mid point of $BC$, bisect $AC$ at $E$ Draw $BQ$ parallel to $PE$ to meet $AC$ at $Q$, then $Q$ is between $E$ and $A$ and $PQ$ is bisector of triangle $ABC$

3 Diagonals $AC$, $BD$ bisect at right angles at $O$

. $PA \cdot PC = PO^2 - AO^2 = PB^2 - AB^2$

4 In inscribed triangle side $= 6 \sin 66° = 5\ 48$ cm

In describing triangle side $= 3 (\tan 73\frac{1}{2}° + \tan 49\frac{1}{2}°) = 13\ 64$ cm

5 See Paper 89, Question 6

$ACDE$ is cyclic . $AD\ \ CE = AE\ \ CD + DE\ \ AC$, etc

Or by congruent triangles

6 $5\ 8^2 - 4\ 2^2 = 10 \times 1\ 6 = 4^2$ · triangle is right angled

Angles are 90°, 46° 24′, 43° 36

## No. 99

1 Triangles $AOB$, $POQ$ are congruent angles $OBA$, $OQP$ are equal $OB = OQ$ angles $OBQ$, $OQB$ are equal Hence angle $VBQ$ = angle $VQB$

2 With centres $C$ and $D$, radii each 2 5 in, strike arcs cutting $DA$ and $DA$ produced at $Q$ and $P$ respectively Then parallelogram $PBCQ$ = parallelogram $ABCD$ At $B$ and $P$ make angles $= 110°$, etc Other sides $= 2\ 47$ in

3 Angle $QEP = 2\,C$, $E$ is centre of circle $APC$

$F$ is centre of circle $AQPB$ angle $QFP = 2^{ce}$ angle $QAP = 180 - 2C$

angles $QEP$, $QFP$ together equal 180°

4 At any point $X$ on $A$ circle draw tangent $XH = 3$ cm

At any point $Y$ on $B$ circle, draw tangent $YK = 4$ cm

Circles centre $A$, radius $AH$, and centre $B$, radius $BK$, intersect at points $P$.

5 Angle $AQB = 180 - ACB$ (cyclic quadrilateral) Angle $PQB = \frac{1}{2}ACB$

. Angle $PBQ = 180 - (180 - ACB) - \frac{1}{2}\,ACB$ $PB = PQ$

6 Let $X$, $Y$, $Z$ be centres of escribed circles touching $BC$, $CA$, $AB$ respectively, and let $I$ be centre of inscribed circle

$ZBCY$ is cyclic quadrilateral angle $XZI$ = angle $XYI$ But $XI$ is common chord of circles $XZI$, $XYI$ and they subtend equal angles at the circumference Hence circles $XZI$, $XYI$ are equal, etc

## No. 100

1 Make angle $BCD = 45° - \dfrac{B}{2}$

2 Triangles $BOA$, $BCD$ are congruent triangles $BEA$, $BED$ are congruent.

3 $ALKB$ is a fixed circle with diameter $AB$ Angle $LAK$ is fixed in magnitude $LK$ is of constant length

4 Bisect $AB$ at $E$, draw $EP$ parallel to $BC$

5 *Reductio ad absurdum*

Angle $DBC = ACB = BAC$ . $DB$ is a tangent.

$DE\ \ DC = DB^2$ · $AB\ \ DE = AC^2$

6 Triangles $OAC$, $ODB$ are congruent, since $A$, $B$ $D$, $C$ are concyclic.

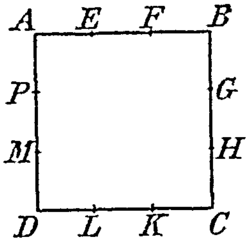
. angle $EAB$ = angle $OBD$ and triangles $EAB$ $OBD$ are similar

. angles $BOF$, $FBO$ are equal and $OF = BF$

MADE IN GREAT BRITAIN AT THE PITMAN PRESS BATH
C6—(1192)

## No. 13

1. In the figure the sides of a square $ABCD$ are each divided into three equal parts, $E$, $F$, $G$, $H$, $K$, $L$, $M$, $P$ being the points of trisection. Prove that $E$, $G$, $K$, $M$ are the corners of a square.




2. In the same figure determine what fraction the area of the octagon $EFGH$ $KLMP$ is of the area of (i) the square $ABCD$, (ii) the square $EGKM$.

3. State and prove the geometrical proposition corresponding to the algebraical formula $a(a+b) = a^2 + ab$.

Four points $A$, $B$, $C$, $D$ are taken in that order on a straight line, show that—

$$AB.CD + AD.BC = AC.BD.$$

4. Show how to draw two tangents to a circle from a point outside it. Prove that the parts of these tangents between the point and the circle are equal.

If the sides $AB$, $BC$, $CD$, $DA$ of a quadrilateral each touch the same circle, prove that $AB + CD = BC + DA$.

5. On a line 2·3 in. long describe a segment of a circle to contain an angle of 70°. Measure the radius.

What is the area of the largest triangle that has base 2·3 in. long and the opposite angle 70° ?

6. $C$ is a fixed point on a diameter $AB$ of a given circle; $PCQ$ is any chord through $C$; at $C$ a perpendicular is drawn to $AB$ which meets $AP$, $AQ$ (produced if necessary) in $R$ and $S$. Prove that $P$, $Q$, $R$, $S$ lie on a circle and that the rectangle $CR.CS$ is of constant area.

## No 14

1 State a construction, with proof, for drawing a line through a given point parallel to a given line

Draw a line 5 5 in long, and by means of parallels divide it into six equal parts

2 In a triangle $ABC$ the side $AB$ is greater than the side $AC$, the bisector of the angle $A$ meets $BC$ at $D$ Prove that the angle $ADB$ is an obtuse angle

3 State and prove a rule for finding the area of a trapezium Construct a trapezium $ABCD$, having $AD$ parallel to $BC$, $AD = 4\ 8$ cm, $BC = 3\ 2$ cm, angle $A = 42°$ and angle $D = 68°$ Find its area

4 Define a circle, and prove that—

(i) A straight line cannot meet a circle in more than two points

(ii) Two circles cannot cut in more than two points

5 If two chords of a circle intersect within the circle, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other

By a geometrical construction find the value of the fraction $\frac{3\ 4 \times 2\ 3}{2\ 8}$, correct to one decimal place Verify by calculation

6 Any point $P$ is taken on the circumference of a circle of which $AB$ is any diameter $PB$ meets the radius $OC$, perpendicular to $AB$, at $R$, and the tangent at $P$ meets the radius $OC$ produced at $Q$ Prove that $QP = QR$

## No. 15

1. Prove that the sum of the diagonals of a quadrilateral is less than the sum of the sides of the quadrilateral, but greater than half the sum of the sides.

2. Draw two triangles to show that two triangles may have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, and yet not be equal in all respects.

The diagonal $AC$ of a quadrilateral $ABCD$ bisects each of the angles $BAD$, $BCD$. Prove that it bisects the diagonal $BD$.

3. Through a point $O$ on the diagonal $AC$ of a parallelogram $ABCD$, $HOF$, $EOG$ are drawn parallel to the sides, $H$ being in $AB$ and $E$ in $BC$. Prove that the parallelograms $HE$, $FG$ are equal in area. Prove also that $HG$ and $EF$ are parallel.

4. If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles supplementary, prove that the triangles are equal in area.

Squares $ABDE$, $ACFG$, $BCHK$ are described on the sides of a triangle $ABC$ with a right angle at $C$. Prove that the triangles $AGE$, $BKD$ are equal in area.

5. Of all lines drawn from a point within a circle to the circumference, which are the greatest and least? Of all chords that can be drawn through a point within a circle which are the greatest and least?

On $OA$ a radius of a circle with centre $O$ and radius 5 cm., a point $P$ is taken 2 cm. from $O$. With $P$ as centre a circle is described with radius 1 cm.; draw the longest and shortest chords of the big circle that will touch the small circle. Prove the accuracy of your construction.

6. A straight line $PQ$ of constant length slides so that $P$ moves along a line $OX$, and $Q$ along a line $OY$. Find the locus of the centre of the circumcircle of the triangle $POQ$.

## No. 16

1 Two isosceles triangles are described on the same base and on the same side of it Prove that the line joining the vertices, when produced, bisects the base at right angles

2 A bar $AB$, 4 ft long is suspended in a horizontal position by ropes $AC$, $BD$, respectively 8 ft and 6 ft long, from two hooks in the ceiling $CD$ 10 ft apart By drawing a figure to scale find the depth of $AB$ below the ceiling

3 Draw a triangle $OAB$ with sides $OA = 2$ in , $OB = 2\ 4$ in , angle $AOB = 35°$ , then draw a triangle $OPQ$ with sides $OP = 2$ in , $OQ = 2\ 4$ in , angle $OPQ = 35°$ and having $OP$ inclined at 45° to $OA$ Verify, and prove by argument, that the angle between $PQ$ and $AB$ is also 40°

4 Give a construction for drawing the tangents to a circle from an external point and justify your construction

If the two tangents drawn from $P$ to a circle with centre $O$ touch the circle at $A$ and $B$ prove that $PO$ bisects $AB$ at right angles

5 State how to find the centre of the circumcircle of a triangle

Perpendiculars $BE$ $CF$ are let fall on the opposite sides of a triangle $ABC$ , prove that the triangles $BFC$, $EFC$ have the same circumcentre

6 From a point $P$ two tangents $PA$, $PB$ are drawn to a circle, and the perpendicular from $P$ to $AB$ meets the circle at $C$ and $D$ Prove that $AC$ bisects the angle $PAB$ and that $AD$ bisects the angle between $AB$ and $PA$ produced

## No. 17

1. The side $BC$ of a triangle is bisected at $D$ and $AD$ is produced to $E$ so that $DE = AD$. Prove that $CE = AB$.

If $AC$ is produced to any point $X$, prove that the angle $BCX$ is greater than the angle $ABC$.

2. Draw a triangle $ABC$ having angle $A = 48°$, angle $C = 63°$, $AB = 7$ cm. In it find a point $P$ such that $PA = PB$, and angle $PAB$ = angle $PBC$.

3. State and prove Pythagoras's theorem.

In a quadrilateral $ABCD$ the angles at $A$ and $B$ are each a right angle, and $AB = AD + BC$. Prove that $CD^2 = 2(AD^2 + BC^2)$.

4. Prove, completely, that the angle at the centre of a circle is double any angle at the circumference standing on the same arc.

Any point $P$ is taken on the circumference of a circle with centre $O$, and is joined to the extremity $A$ of a diameter $AB$. If $PA$ is produced to $Q$, prove that the angle $BAQ$ is half the reflex angle $BOP$.

5. Prove that the perpendicular from the centre of a circle to any chord bisects that chord.

$A$ and $B$ are any two points on a circle, show how to draw two parallel chords $AP$, $BQ$ such that $AP = 2BQ$.

6. An equilateral triangle $ABE$ is described on the side $AB$ of a square $ABCD$ and on the same side of it as the square. The line from $A$ at right angles to $DE$ meets $CD$ at $F$. Prove that a circle can be described about $AEFD$.

Show also that $DF = CD(2 - \sqrt{3})$.

## No 18

1 On the sides $AB$ $AC$ of a triangle equilateral triangles $ABE$ $ACD$ are described outside the triangle $ABC$ Prove that $BD = CE$

2 If three parallel straight lines cut off equal intercepts on one transversal prove that the intercepts on any other transversal are also equal

Two straight lines $AX$ $AY$ contain an angle 55° a point $P$ is taken within the angle $XAY$ 1 5 cm from $AX$ and 2 cm from $AC$ Construct a straight line through $P$ so that the part intercepted between $AX$ and $AY$ is bisected at $P$

State and prove your construction

3 Prove that a diagonal of a parallelogram always bisects the area but in general does not bisect the angles of the parallelogram

In a trapezium $ABCD$ the sides $AB$ and $DC$ are parallel and $AB$ is less than $DC$ Through $A$ a line is drawn parallel to $BC$ meeting $DC$ at $E$ and $DE$ is bisected at $F$ Prove that the triangle $BCF$ is half the trapezium

4 *The opposite sides $AB$ and $DC$ of an irregular quadri* lateral meet when produced at $E$ The circumcircles of the triangles $BEC$ $ADE$ meet again at $F$ Prove that the angles $BFC$ $AFD$ are equal

5 At the ends $A$ and $B$ of a diameter of a circle tangents are drawn these tangents are cut by a third tangent at $C$ and $D$ respectively Prove that the rectangle $AC$ $BD$ is equal to the square on the radius of the circle

6 Divide a straight line $AB$ 6 cm long into two parts at a point $P$ so that the rectangle contained by the whole line $AB$ and the part $BP$ shall equal the square on the part $AP$ State your construction and prove its validity Also draw any circle to pass through $A$ and $P$ from $B$ draw a tangent $BQ$ to this circle With centre $B$ and radius $BQ$ draw a circle cutting $BA$ at $R$ Prove that $AR = BP$

## No. 19

1. The diagonal $DB$ of a parallelogram $ABCD$ is produced to $E$, so that $BE = DB$, and the parallelogram $CBEF$ is completed. Prove that $AB$ and $BF$ are in the same straight line.

2. In a triangle $ABC$ with a right angle at $C$, any point $D$ is taken in $BC$, and any point $E$ in $AC$; prove that $DE$ must be less than $AB$.

Two right-angled triangles $ABC$, $DEF$ have their hypotenuses $AB$ and $DE$ equal, but $AC$ is less than $DF$. Prove that angle $ABC$ is less than the angle $DEF$.

3. Construct a quadrilateral $PQRS$, having $PQ = 2{\cdot}4$ cm., the diagonal $QS = 3{\cdot}2$ cm., angle $QPS = 66°$, angle $SQR = 39°$, and angle $QSR = 68°$.

Make a triangle equal to it, and test the accuracy of your construction by calculating both areas.

4. What is meant by the *projection* of a finite straight line on another?

Show that in any triangle the difference of the squares on any two sides of a triangle is equal to twice the rectangle contained by the third side and the projection on it of the median bisecting that side.

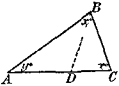
5. Write out two enunciations in which the word *alternate* is used, and explain the meanings of the word.

The railway line between two places X and Y consists of two circular arcs, $XZ$ and $ZY$, which have a common tangent $PZQ$, the arcs being continuous and on opposite sides of the tangent. Draw a plan of the line, scale 1 cm. to the mile, being given that the chord $ZZ$ is 3 miles long, the angle $XZP$ is 30°, the angle $YZQ$ is 25° and the radius of the arc $YZ$ is 5 miles. What is the shortest distance from X to Y?

6. $P$ and $Q$ are two points $4{\cdot}5$ in. apart; draw a straight line such that the shortest distance from $P$ to it shall be 3 in., and from $Q$ 2 in.

## No 20

1 If an isosceles triangle $ABC$ has angles $x°$ and $y°$ as in the figure, determine the value of $x$ and $y$ so that it may be possible to draw a line $BD$ to divide the triangle into two isosceles triangles $ADB$, $CDB$ Prove that $AD$ is greater than $DC$




2 Two points $A$ and $B$ are on the same side of a straight line $XY$, at unequal distances from it Show with proof how to find a point $C$ in $XY$ such that the angle $ACX$ = the angle $BCY$

If $P$ be any other point in $XY$, prove that—

$$AC + CB < AP + PB$$

3 Prove that the area of a triangle is one half the area of the rectangle on the same base and with the same altitude

On the side $AC$ of a triangle $ABC$ right angled at $C$, an equilateral triangle $ADC$ is described Prove that the triangle $BCD$ is half the triangle $ABC$

4 Show that four circles can be drawn to touch three straight lines each of which cuts the other two

The centre of the inscribed circle of a triangle $ABC$ is $I$, and the centre of the circle touching $BC$ and the sides $AB$ and $AC$ produced is $E$ Prove that the circle described on $IE$ as diameter passes through $B$ and $C$

5 A point $D$ is taken in a side $BC$ of an equilateral triangle $ABC$, and an equilateral triangle $CDE$ is described on $CD$, the vertices $A$ and $E$ being on opposite sides of $BC$, and $AD$ is produced to meet $BE$ at $F$ Prove that the circumcircle of the triangle $BDF$ touches $AB$

6 Prove that the perpendiculars from the vertices of a triangle upon the opposite sides meet in a point

## No. 21

1. Prove that two right-angled triangles are congruent if they have (i) the hypotenuse and an acute angle of the one equal to the hypotenuse and an acute angle of the other, or (ii) the hypotenuse and another side of the one equal to the hypotenuse and another side of the other, each to each.

2. Draw the locus of all acute-angled triangles on a base 3·5 cm., having area 7 sq. cm.

In the same figure draw the particular triangle that has the smallest perimeter and satisfies the conditions. Give your reasons.

3. Explain how to construct a square equal to a given rectangle.

Divide a straight line of length 4·3 cm. internally, so that the rectangle contained by the segments is of area 3 sq. cm. State your construction, which must be entirely geometrical, and not depend on an arithmetical extraction of a square root.

4. State, with proof, how to find the centre of a given circular arc.

Find, to the nearest millimetre, the radius of the arc in the adjoining figure.

5. If the opposite angles of a quadrilateral are supplementary, prove that a circle can be described to pass through its angular points.

The vertex $A$ of an equilateral triangle $ABC$ is joined to a point $D$ on $BC$ produced, and on $AD$ an equilateral triangle $ADE$ is described. Prove that either $A$, $B$, $D$, $E$, or $A$, $C$, $D$, $E$, are concyclic.

6. Two circles intersect, one of the points of intersection being $P$. Explain, with proof, how to draw through $P$ a line $QPR$, meeting one circle at $R$ and the other at $Q$, such that $QR$ is bisected at $P$.

## No 22

1 From two points $A$ and $B$ on a straight shore, 150 yds apart, a rock $C$ is seen and the angles $CAB$, $CBA$ are 45° and 60° respectively Find, by drawing to scale 1 cm = 20 yds, and without using a protractor, the distance of $C$ from the shore

2 Prove, by the theory of parallels that the sum of the angles of a triangle is equal to two right angles

State the similar fact about a rectilinear figure of $n$ sides

Draw any irregular pentagon, having each of its angles obtuse, produce the sides of the pentagon to meet, thus forming a five pointed star Find the sum of the angles at the points of the star

3 The side $BC$ of a parallelogram $ABCD$ is bisected at $E$, and the side $CD$ at $F$ Prove that the area of the triangle $AEF$ is three times the area of the triangle $CEF$, and that together they are equal to half the parallelogram

4 In any circle prove that equal chords are equi distant from the centre

A point $P$ is taken 2 3 in from the centre $O$ of a circle with radius 1 5 in Draw through $P$ a line meeting the circle at $A$ and $B$ so that the length of $AB$ shall be 1 9 in Explain your construction

5 Explain how to draw a circle to pass through two given points and to touch a given straight line

6 $ABC$ is an isosceles triangle in which the angles at $B$ and $C$ are each twice the angle at $A$, at $C$ an angle $BCD$ is made equal to the angle at $A$, $CD$ meeting $AB$ at $D$ Calculate the size in degrees of all the angles in the figure, and prove that rectangle $BA$, $BD$ = square on $AD$

## No. 23

1. State with proof, the construction for bisecting with ruler and compass a given angle.

Without using a protractor draw an angle of 75° and divide it into five equal parts.

2. If a quadrilateral has both pairs of opposite sides equal to one another, prove that its diagonals bisect one another.

3. State and prove the proposition concerning the square on the side of a triangle opposite an obtuse angle.

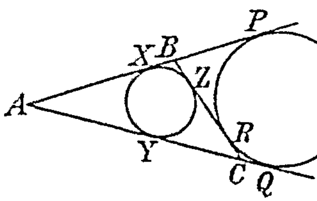
A triangle has sides 1·7 in., 1·3 in., 1·1 in., determine, firstly by drawing, secondly by calculating, whether the centre of the circumcircle is inside or outside the circle.

4. Prove that equal arcs in a circle subtend equal angles at the centre, and that equal chords subtend equal angles at the centre.

If an arc $AB$ is double an arc $PQ$ in the same circle, with centre $O$, prove that angle $AOB$ is double the angle $POQ$; but if in a circle a chord $AB$ is double a chord $PQ$, the angle $AOB$ is more than double the angle $POQ$.

5. In a quadrilateral $ABCD$ the diagonals $AC$, $BD$ intersect at $E$. Construct the quadrilateral being given $AE = 2$ cm., $BE = 1$ cm., $CE = 1·5$ cm., angle $ABD = 100°$, and angle $ACD = 100°$.

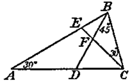
6. In the figure the two circles touch the two lines $AB$, $AC$ at $X$ and $Y$, $P$ and $Q$; and $BC$ is an internal common tangent to the two circles touching at $Z$ and $R$. Prove that—




(i) $2AP = AB + BC + CA$.
(ii) $2AX = AB + AC - BC$.
(iii) $ZR = AC - AB$.

## No. 24

1 In the figure $AB = AC$ and the angles $CAB$, $CBF$, $BCF$ have the values shown Prove that $DA = DB$ and $BE = BF$.




2 Define a parallelogram, and prove that its opposite sides are equal

A pair of parallel lines $AB$ and $DC$ intersect another pair of parallel lines $AD$ and $BC$ at the points $A$, $B$, $C$, $D$ A point $P$ is taken anywhere in the plane of these lines Show, with proof, how to draw through $P$ a line such that the part intercepted between one pair of parallel lines is equal to the part intercepted between the other pair

3 The area of a quadrilateral is 24 sq in, and its two diagonals are 6 in and 8 in in length Prove that the diagonals are at right angles, and that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair

4 Prove that if two circles touch, whether internally or externally, the point of contact lies on the line joining the centres

Draw two circles with radii 4 cm and 5 cm, and with their centres 7 cm apart Draw a third circle with radius 3 cm to touch each of the other two

5 Prove that the angles between a tangent to a circle and a chord through the point of contact are equal to the angles in the alternate segments

Tangents are drawn at two points $A$ and $B$ on a circle, and from a third point $P$ on the circle perpendiculars $PH$, $PK$ are let fall on the tangents and $PL$ on the chord $AB$ Prove that the triangles $PHL$, $PKL$ have their angles equal, each to each

6 Construct, without any long calculation of square roots, a triangle whose sides are $\sqrt{5}$ cm, $\sqrt{7}$ cm, $\sqrt{12}$ cm, and measure the largest angle

## No. 25

1. Construct a triangle with sides $AB = 4$ cm., $BC = 3$ cm., $AC = 5$ cm. Draw a straight line $PQ$ 6 cm. long; construct, without using a protractor, an angle $PQR$ equal to the angle $ACB$ and an angle $QPR$ equal to angle $BAC$. Measure $PR$ and $QR$.

2. The middle points $D$, $E$, $F$ of the sides $BC$, $CA$, $AB$ of a triangle are joined. Prove that $AD$ is bisected by $FE$ and that the area of the triangle $DEF$ is one-quarter of the area of the triangle $ABC$.

3. Give the enunciation which states by how much the square on the side subtending an acute angle exceeds the sum of the squares on the sides containing that angle.

In a triangle $ABC$, $AD$ is the median bisecting $BC$, and $AX$ is at right angles to $BC$; prove the difference between the squares on $AB$ and $AC$ is equal to twice the rectangle $BC \cdot DX$.

Write out the general enunciation corresponding to this particular enunciation.

4. Prove, fully, that the middle point of the hypotenuse of a right-angled triangle is equidistant from the three angular points.

$AB$ is a diameter of a circle of radius 3·7 cm., and $AC$ is a chord 2 cm. long. Without any calculation, construct a rectangle equal in area to the square on $AC$ and having one side equal to $AB$. Give a proof.

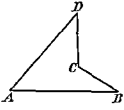
5. In a circle, centre $O$, radius 1·5 in., inscribe a quadrilateral $ABCD$ having $BC = CD$, the angle $ABC = 95°$ and the angle $AOB = 40°$. State the steps of your construction.

Calculate the number of degrees in the angle $BCD$, and so check the accuracy of your figure.

6. Find the locus of a point which moves so that the tangents drawn from it to two fixed circles are equal. [Consider separately two cases, (i) the circles intersect, (ii) the circles do not intersect.]

## No 26

1 Copy the quadrilateral $ABCD$ by *measuring* any *two* angles and as few lengths as possible Give the enunciations of the congruency propositions that justify your construction




2 A triangle $ABC$ has the sides $AB$ and $AC$ equal, the angles at $B$ and $C$ are bisected by lines meeting the opposite sides at $P$ and $Q$ Prove that $PQ$ is parallel to $BC$

3 Prove that a parallelogram and rectangle on the same base and between the same parallels are equal in area

Squares $AGFC$, $BCDE$ are described on the sides $AC$, $BC$ of a *triangle right angled at $A$ $DC$ is produced to meet $FG$ pro* duced at $H$, and a line through $A$ parallel to $DH$ meets $ED$ at $K$ and $FG$ produced at $L$ From this figure prove that the square $AF$ = rectangle $CK$

4 Prove that any two angles at the circumference of a circle standing on the same chord are either equal or supplementary

$ABCD$ is a cyclic quadrilateral and $X$ is the middle point of the arc $BD$ on the side of $CD$, remote from $A$ Prove that $XC$ bisects the angle between $DC$ and $BC$ produced to $E$

5 Two circles with centres $P$ and $Q$ touch externally at $A$, an external common tangent touches the circles at $B$ and $C$ respectively, and meets the common tangent drawn through $A$ at $D$ *Prove that the angles $BAC$ and $PDQ$ are each equal* to a right angle

6 Show, with proof, how to produce a chord $AB$ of a circle to a point $P$, such that rectangle $AP \cdot PB$ = square on $AB$

## No. 27

1. Two triangles $ABC$, $ADE$ have $AB = AD$, $AC = AE$, $BC = DE$, and are placed so that the line $CE$ cuts both $AB$ and $AD$ without being produced. Prove that $CE$ is parallel to $BD$.

2. The angles $A$, $B$, $C$ of a quadrilateral $ABCD$ are respectively 100°, 70° and 150°. A new quadrilateral is formed by bisecting the exterior angles of $ABCD$. Find the sizes of the interior angles of this new quadrilateral.

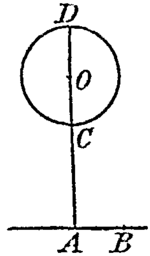
3. If the square on one side of a triangle is equal to the sum of the squares on the other two sides, prove that the triangle is right-angled.

Find the area of a triangle whose sides are 17·8 in., 16 in., 7·8 in.

4. Write the enunciation connecting the squares on the sides of a triangle with the square on one of the medians.

In a triangle $ABC$, the angle $C$ is a right angle, and $AD$ and $BE$ are medians. Prove that $4AD^2 + 4BE^2 = 5\,AB^2$.

5. Reproduce the adjoining figure taking $OA = 3$ in., $AB = 1$ in., and radius of circle to be 1 in.




Construct two circles each of which touches the given circle, and also touches the line $AB$ at $B$. Prove your construction and measure the distance between the centres.

6. The diagonals of a trapezium $ABCD$, in which $AB$ is parallel to $CD$, intersect at $E$; prove that the circles described about the triangles $ABE$, $CDE$ touch. If the circles described about the triangles $ADE$, $BCE$. also touch, prove that $ABCD$ must be a parallelogram.

## No. 28

1 A line $AB$ of definite length is moved to any other position $A'B'$ in the same plane, the perpendicular bisectors of $AA'$ and $BB$ meet at $O$ Prove that the angle $AOB$ is equal to the angle $A'OB'$

2 Show that parallelograms on equal bases and between the same parallels are equal in area

$ABCD$ is a parallelogram, equal lengths $AP$, $DQ$ are cut off from the sides $AB$, $DC$ Any two parallel lines are drawn through $P$ and $Q$, show that the parallelogram they intercept between $AD$ and $BC$ is always of the same area as $ABCD$

3 In a triangle $ABC$, a perpendicular $AD$ is drawn from $A$ to $BC$ If $AB = 65$ cm, $AD = 60$ cm, $AC = 156$ cm Show that the triangle is right-angled

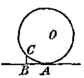
4 If two circles intersect prove that the line joining their centres, produced if necessary, bisects the common chords at right angles

Being given a circle of radius 4 cm, draw two other circles of radius 2 cm so that they intersect the first circle in the same two points the common chord being of length 3 cm

5 Prove that a line passing through the centre of a circle perpendicular to a chord bisects the chord

The longest line that can be drawn from a point $P$ inside a circle to the circumference is 8 cm long the shortest is 2 cm Construct the figure Measure the length of the chord that is bisected at $P$, and verify by calculating the length of that chord

6 The figure represents a wheel of radius 2 ft 1 in, which is just about to roll, without slipping, over a thin obstacle $BC$ of height 10 in Calculate the length of $BA$




In a figure drawn to the scale of 1 cm = 10 in, trace out the loci described by $O$ and $A$ respectively, while the wheel is surmounting the obstacle

## No. 29

1. Prove—

(i) The vertex of an isosceles triangle lies on the perpendicular bisector of the base.

(ii) All points equidistant from two fixed points lie on the perpendicular bisector of the line joining the points.

(iii) The centres of all circles passing through two given points lie on the perpendicular bisector of the line joining the points.

2. A line $PQ$ is bisected at $R$; from $P$, $Q$, and $R$ perpendiculars $PH$, $QK$, $RL$ are let fall on another straight line passing through a point $O$, not between $H$ and $K$. Prove that (i) $2OL = OH + OK$, (ii) $2RL = PH + QK$.

If a line $AB$ is divided equally at $O$ and unequally at $P$, prove (i) geometrically, (ii) algebraically, that $OP$ is equal to the difference between $AP$ and $PB$.

3. Construct a quadrilateral $ABCD$ having $AB = 2{\cdot}3$ cm., the diagonal $BD = 3$ cm., angle $BAD = 65°$, angle $DBC = 40°$, and the angle $BDC = 70°$. Make a triangle equal to it, and prove your construction to be correct.

Making the necessary measurements, calculate the areas of the quadrilateral and triangle.

4. $AOB$ is a straight line bisected at $O$; a point $P$ moves so that $AP^2 + BP^2$ is constant. Prove that the locus of $P$ is a circle.

Draw the circle when $AB = 3$ in. and the constant is $8\frac{1}{2}$ sq. in.

5. Any point $D$ is taken in the side $AB$ of a triangle $ABC$; through $D$ a line is drawn, meeting $AC$ at $E$ and making the angle $ADE$ equal to the angle $ACB$. Prove that the rectangle $AC \cdot AE$ is equal to the rectangle $AD \cdot AB$.

6. Without using a protractor, construct on a base $BC$ 1 in. long an isosceles triangle $ABC$ having the angle $A$ equal to 36°.

## No 30

1 If two triangles have two sides of one equal, each to each, *to two sides of the other, but the angle contained by the two* sides of the first triangle greater than the angle contained by the corresponding sides of the other, prove that the third side of the first is greater than the third side of the other

Two triangles $ABC$, $AED$ have $AD$, $AC$ in the same straight line, and $AD$ less than $AC$, $AB = AE$, and the angles at $D$ and $C$ are right angles but on opposite sides of $AC$ Prove that the angle $DAE$ is greater than the angle $CAB$

2 What do you know about the line joining the middle points of two sides of a triangle ?

The base $BC$ of an isosceles triangle $ABC$ is bisected at $D$ and $AC$, one of the equal sides, at $E$, the lines $BE$, $DE$ are bisected at $H$ and $K$ respectively Prove that $DH = DK$ and $AH = AK$

3 Construct a rhombus of side 2 5 in equal in area to a square of side 2 in

Check your figure by measuring the diagonals of the rhombus and calculating the area

4 On a base 2 5 in long construct a segment of a circle to contain an angle of 100°

A cyclic quadrilateral $ABCD$, the diagonals of which intersect at $O$, is such that $AOB = 140°$ $AB = 2\ 5$ in $BC = 1$ in, $AC = 2\ 5$ in Construct the quadrilateral and measure $CD$.

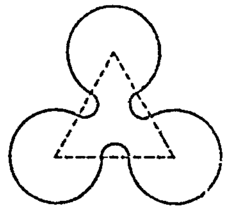
5 A point $P$ is taken 4 cm from the centre of a given circle with radius 2 5 Draw through $P$ a straight line which will cut from the given circle a segment containing an angle of 100°

6 The perpendicular let fall from the vertex $A$ of a triangle $ABC$ to the side $BC$ meets $BC$ at $D$, and the circumcircle at $E$ the perpendicular from $B$ to $AC$ meets $AD$ at $K$ Prove that $KD = DE$

## No. 31

1. Two straight lines $AB$, $CD$ bisect each other at $O$; $AC$, $BD$ are bisected at $E$, $F$ respectively. Prove that $EF$ passes through $O$.

2. Draw an equilateral triangle with side 4 cm., and then draw accurately a figure similar to the adjoining figure.



3. By a geometrical construction obtain an approximate solution of the equations $x + y = 10$, $xy = 20$. Explain and justify your method.

4. (i) $A$, $B$, $C$, $D$ are four points on the circumference of a circle. If $AC$ and $BD$ bisect one another, prove that they both pass through the centre.

(ii) A diameter $AB$ of a circle bisects a chord $CD$. If $BC$ is parallel to $AD$, prove that $CD$ bisects $AB$.

5. If a line subtends equal angles at two points on the same side of it, prove that the two points and the extremities of the line lie on a circle.

$AB$, $BC$, $CD$ are three straight lines, of which $AB$ and $CD$ are equal and on the same side of $BC$, and the angle $ABC$ is equal to the angle $BCD$. Prove that the four points $A$, $B$, $C$, $D$ lie on a circle. Prove also that $AD$ is parallel to $BC$.

6. A circle is described about an equilateral triangle $ABC$, and another equilateral triangle $ABD$ is described on the other side of $AB$; with centre $D$ and radius $DA$ a circle is described. A point $P$ is taken on the circumference of the first circle; $PA$ and $PB$, produced if necessary, meet the second circle at $Q$ and $R$ respectively. Prove (i) that $QR$ is a diameter of the second circle, (ii) that $S$ the intersection of $AR$ and $BQ$ lies on the first circle.

## No 32

1 $ABCD$ is a rectangular croquet lawn the width $AD$ being 40 ft On the lawn are three balls $P$ $Q$ $R$ $P$ is 10 ft from $AB$ and 5 ft from $AD$ $Q$ is 20 ft from $A$ and 30 ft from $D$ $R$ is equidistant from $P$ and $Q$ and also equidistant from $AD$ and $AB$

Draw a figure to a convenient scale and so find the distances of $P$ from $AB$ and $AD$

2 Define *parallel* straight lines Give without proof a practical test for finding whether two given lines may be considered parallel

If all the angles of a hexagon are equal prove that three pairs of sides are parallel

3 What is meant by the *orthocentre* of a triangle? Through the angular points of a triangle $ABC$ lines are drawn parallel to the opposite sides these parallels forming a triangle $PQR$ Prove that the orthocentre of $ABC$ coincides with the circum centre of $PQR$

4 $AB$ is a chord of a circle centre $O$ the perpendicular bisector of $OB$ meets $AB$ at $C$ and $CO$ is produced to any point $D$ Prove that the angle $AOD$ is three times the angle $BOC$

5 State and prove a construction for drawing a circle to touch the side $BC$ of a triangle $ABC$ and the sides $AB$ $AC$ produced

If this circle touches $AB$ and $AC$ produced at $P$ and $Q$ respectively prove that $AP = s$ when $s$ is the semiperimeter of the triangle $ABC$

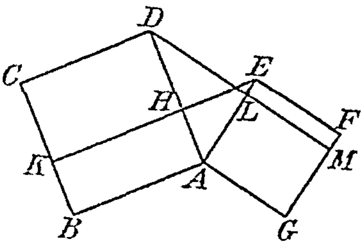
6 An endless belt passes round two wheels one 5 ft and the other 2 ft in radius their centres being 9 ft apart By drawing an accurate figure to the scale 1 cm = 1 ft find the length of the portions of the belt (assumed to be straight) between the wheels Verify this by calculation

## No. 33

1. $BAC$ is any angle and $AB = AC$. From $AB$, $AC$ respectively equal lengths $AP$, $AQ$ are cut; $CP$ and $BQ$ intersect in $X$. Prove that $AX$ bisects the angle $BAC$.

2. Draw a trapezium $ABCD$, being given that $AB$ and $DC$ are parallel and $AB = 2{\cdot}4$ in., $BC = 3{\cdot}6$ in., $CD = 4{\cdot}5$ in., $DA = 3$ in. State the steps of your construction.

3. Write out in full the enunciations which connect the square on one of the sides of a triangle with the squares on the other sides.




Two squares $ABCD$, $AGFE$ are placed as in the figure; $EHK$ is parallel to $AB$ and $DLM$ is parallel to $AG$. Prove that the rectangles $ABKH$ and $AGML$ are equal, (i) by proving them to be double equal triangles, (ii) by using one of the propositions mentioned in the first part of this question.

4. Show that a square is the only parallelogram which can be inscribed in a circle and also have a circle inscribed in it.

5. If from a point $P$ outside a circle a tangent $PA$ and a secant $PBC$ are drawn, show, without using the properties of similar figures, that $PA^2 = PB \, . \, PC$.

A point $P$ is taken on the produced common chord of two circles; $PT$ is drawn to touch one of the circles and $PQR$ is a secant of the other. Prove that the circumcircle of the triangle $QRT$ touches one of the original circles.

6. Draw a circle of $2{\cdot}3$ in. radius and take a point $A$ on the circumference. Inscribe a triangle $ABC$ in the circle having $A = 60°$, and $B = 75°$.

Give reasons for your construction and measure the sides of the triangle.

## No. 34

1 In a triangle $ABC$ the side $AB$ is greater than $AC$, the bisector of the angle $A$ meets $BC$ at $D$ Prove that $BD$ is greater than $DC$

2 Prove that triangles of equal area, on the same base and on the same side of it, lie between the same parallels

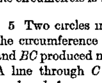
Draw the complete locus of the vertices of all acute angled triangles having a given $BC$ 6 cm long for base, and area 6 sq cm Give your reasons

3 The diagonal $AC$ of a parallelogram $ABCD$ is produced to $E$ so that $AE = 2AC$, and $DC$ is produced to meet $EB$ at $F$ Prove that $F$ is the mid point of $BE$ and that $DC = 2CF$

4 Construct a triangle $ABC$ being given that the radius of the circumcircle is 1 5 in, $BC = 2$ in, and $CA = 1$ 2 in

5 Two circles intersect at $A$ and $B$, a point $C$ is taken on the circumference of one of the circles inside the other, $AC$ and $BC$ produced meet the second circle at $D$ and $E$ respectively A line through $C$ parallel to $DE$ meets $AE$ at $P$, and $EA$, produced if necessary, meets the first circle at $F$ Prove that the square on $PC$ is equal to the rectangle $PA$, $PF$

6 In the figure $PQRSTVWX$ represents a regular octagon inscribed in the square $ABCD$, prove that $XR = PS$




Hence, or otherwise, find by a geometrical construction the side of a regular octagon inscribed in a square of side 4 cm

If $x$ in is the side of a regular octagon inscribed in a square of side $a$ in, prove that $x(1 + \sqrt{2}) = a$

## No. 35

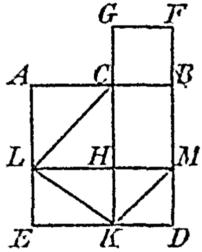
1. From the ends of a finite straight line $AB$ equal parts $AC$ and $BD$ are cut. Through $A$ and $D$ parallel lines $AE$ and $DF$ are drawn; through $C$ and $B$ parallel lines $CE$ and $BF$ are drawn, meeting the other parallels in $E$ and $F$ respectively. Prove that $AF$ and $DE$ bisect one another.

2. Construct a triangle $ABC$ having $BA = 2{\cdot}6$ in., $AC = 3{\cdot}2$ in., angle $ABC = 55°$. Measure the angle $BCA$. Why are there not two solutions?

Also construct and measure the perpendicular from $A$ on $BC$ and calculate the area of the triangle $ABC$.

3. State and prove the geometrical theorem corresponding to the algebraic formula $(a-b)^2 = a^2 - 2ab + b^2$.

In the figure $AH$, $AD$, $BG$ are squares. Prove that the area of the pentagon $GMKLC$ is equal to the rectangle $AC{\cdot}CB$ together with half the sum of the squares on $AC$ and $CF$.




4. Construct a cyclic quadrilateral being given that $AB = 4$ cm., $BC = 6$ cm., $A = 95°$, $B = 110°$.

5. From a point outside a circle two straight lines are drawn, one of which cuts the circle and the other meets it. What is the condition that the meeting line is a tangent? Give a proof.

In a triangle $ABC$ angle $A = 80°$ and angle $C = 40°$. At $A$ a line $AD$ is drawn making angle $CAD = 60°$ and meeting $CB$ at $D$. Prove that $CA^2 = CB, CD$.

6. A circle is described about a triangle $ABC$, and $D$ is the middle point of the arc $BC$, being on the opposite side of the chord $BC$ to $A$. If $I$ is the centre of the inscribed circle, prove that $DI = DB = DC$.

## No. 38

1 Show how to bisect a reflex angle

Two lines $AB$, $AC$ meet at $A$ Prove that the bisectors of the two angles at $A$ (the reflex and the ordinary) are in the same straight line

2 Prove that the point of concurrence of the medians of a triangle is one third of the way from the bisected side to the opposite vertex

The side $AD$ of a parallelogram $ABCD$ is bisected at $E$ and $CE$ meets the diagonal $BD$ at $F$ Prove that $DF$ is one third of $DB$

3 A church $A$ is 3 miles west of another church $B$ A school is to be built so as not to be more than 2 miles from either church Draw a plan and mark the area in which the school may be built If it is as far as possible from both churches calculate its distance from the straight road $AB$, ssing by both churches

4

point Prove that the line joining a point $P$ to the centre $O$ of a

The bisects the angle between the tangents drawn to the

perpe from $P$

on the e points $A$, $B$ $C$ are taken on the circumference of a

5 Pro ich that the angle $ABC$ is 123° Calculate the number

intersect es in the angle between the tangents at $A$ and $C$

Perpen quadrilateral $ABCD$ is inscribed in a circle of radius

opposite s $A$ is the tangent at $A$ Construct the figure being

the line jo angle $EAD = 25°$ angle $ACB = 35°$, and angle

$CK$ is perpe Calculate the angle between the diagonals and

accuracy of your figure

√6 Construc

angle $BAC =$ point $P$ inside a triangle $ABC$, perpendiculars

section of the re let fall on the sides $BC$, $CA$, $AB$ respectively

$+ BD^2 + CE^2 = AE^2 + BF^2 + CD^2$

at the circles described about the triangles

meet in a point

## No. 39

1. In a certain town the railway station $A$ is 200 yds. east of a picture palace $B$; a church $C$ is north-east of $B$ and 250 yds. from $A$. Draw a plan showing the relative positions of $A$, $B$ and $C$, and find the site for a house which is to be equidistant from $A$, $B$, and $C$.

2. In a certain pentagon three of the angles are each double each of the other two angles, find the number of degrees in each angle.
Write out in full the general enunciation you have used.

3. At the middle point $O$ of a straight line $AB$, a perpendicular $OC$ is drawn equal to $OA$; $AC$ and $BC$ are joined From any point $P$ in $OB$ a perpendicular is drawn to $AB$, meeting $BC$ at $D$. $DE$ is drawn parallel to $PO$ to meet $OC$ at $E$ and line $AD$ is drawn. Use this figure to prove—

$$AP^2 + PB^2 = 2AO^2 + 2OP^2.$$

4. Show that four circles can be drawn to touch three straight lines which are not concurrent, and no two of which are parallel.
Show that any one of the centres is the orthocentre of triangle formed by the other three centres.

5. In a circle of radius 3·5 cm. inscribe a triangle $ABC$ having $A = 48°$, $B = 62°$. If the minor arc $BC$, $CA$, $AB$ are bisected at $A'$, $B'$, $C'$ respectively, measure and calculate the sizes of the angles $A'$, $B'$, $C'$.

6. $ABC$ is an equilateral triangle and $P$ a point inside it, on $AP$ an equilateral triangle $APQ$ is described, $P$ and $Q$ being on opposite sides of $AC$. $BP$, $CQ$ are produced to meet at $X$. Prove that $A$, $P$, $Q$, $X$ lie on a circle, and that the line joining the circumcentres of the triangles $ABC$, $APQ$ is at right angles to $AX$.

## No. 40

1. The bisectors of the opposite angles $A$ and $C$ of a parallelogram $ABCD$ meet the diagonal $BD$ at $E$ and $F$ respectively, prove that $AE$ is parallel to $CF$

2 The medians of a triangle $ABC$ intersect at a point $G$ Prove that the triangles $BGC$, $CGA$, $AGB$ are equal in area

3 Draw, using a protractor, a regular pentagon $ABCDE$, and draw an isosceles triangle equal to it in area having $D$ as a vertex Prove your construction to be correct

4 Two concentric circles are drawn, points $A$, $B$ are taken on the inner circle and points $C$, $D$ on the outer so that the angle $OAB$ is equal to the angle $OCD$ Prove that $AC$ is equal to $BD$

5 Perpendiculars $AD$, $BE$, $CF$ are drawn from the vertices of a triangle $ABC$ to the opposite sides, $P$ is the middle point of the side $BC$ Prove that $D$, $E$, $F$, $P$ lie on a circle

6 Prove that the perpendicular bisector of the chord of a circle passes through the centre

Two chords $AB$, $CD$ of a circle with centre $O$ intersect at right angles, the chords $AD$, $BC$ are bisected at $P$ and $Q$ respectively Prove that $OP = CQ$

## No. 41

1. $ABQP$ are four points in order on a straight line such that $AB = PQ$; $ABC$, $PQR$ are equal angles, and $BC = QR$. Prove that $AR = PC$.

2. On a certain public common three paths intersect so as to form a triangle $ABC$. It is found that $AB = 120$ yds., the angle $BAC = 48°$, and the angles $ABC$ and $ACB$ are equal. There is a drinking-fountain at $D$ equidistant from the three paths, but $B$ and $D$ are on opposite sides of $AC$. Draw a plan on the scale of 1 cm. = 30 yds., and explain your construction.

3. Prove that triangles on equal bases and of the same altitude are equal in area.

In a quadrilateral $ABCD$ the diagonals intersect at $E$; $EB$ is produced to $F$ so that $EF = DB$ and $EC$ is produced to $G$, so that $EG = AC$. Prove that the triangle $EFG$ is equal to the quadrilateral $ABCD$.

4. If two circles cut, prove that the common chord is bisected by the line joining the centres, produced if necessary.

$A$ and $B$, distant 6 cm., are the centres of two circles which have a common chord $PQ$, 2·5 cm. long What are the loci of $P$ and $Q$? Draw the two circles, being given that the tangents from $B$ to the other circle are each of length 3·5 cm.

5. Perpendiculars $AX$, $BY$, $CZ$ are let fall from the vertices of a triangle $ABC$ to the opposite sides. Prove that $AX$ bisects the angle $YXZ$.

6. If two chords of a circle intersect, prove that the rectangles contained by their segments are equal.

A triangle $ABC$, with angle $A = 60°$, is inscribed in a circle. A point $P$ is taken on the circumference between $A$ and $B$ and a line $PQRS$ is drawn, cutting $AB$ in $Q$, $AC$ in $R$ and the circle again in $S$, so that the angle $AQR$ is also 60°. Prove that the difference between $AB$ and $AC$ is equal to the difference between $PR$ and $QS$.

## No 42

1 Construct a triangle $ABC$ in which $AB = BC$, $B = 80°$ and $BD$ the perpendicular from $B$ to $AC = 1\ 5$ in

Produce $AB$ to $E$ making $BE = AB$ Join $CE$ Prove that $AE^2 = AC^2 + CE^2$ and verify by measurement and calculation

2 Four points $A$ $B$ $C$ $D$ are taken in that order on a straight line so that $AC = BD$ $EF$ is any straight line parallel to $AD$ Prove that the quadrilaterals $ACFE$ and $BDFE$ are equal in area

3 Prove that in an acute angled triangle the square on any side is less than the sum of the squares on the other two sides by twice the rectangle contained by either of those sides and the projection of the other on it

If the perpendicular from $A$ on the opposite side $BC$ meets it at a point $D$ such that $BD = 2DC$ prove that $AB^2 = AC^2 + BC\ CD$

4 From a point $P$ inside a circle three equal lines $PA$ $PB$ $PC$ are drawn to the circumference show that $P$ must be the centre of the circle

√5 Describe a triangle $ABC$ in which $BC = 2\ 3$ in $B = 40°$ and the radius of the inscribed circle is 1 in State the steps of your construction

6 Explain how to draw two common tangents to two intersecting circles

Prove that the common chord when produced bisects the portion of either common tangent between the points of contact

## No. 43

1. A point $P$ is taken inside a triangle $ABC$, prove that $PB + PC < AB + AC$, and that $\angle BPC > \angle BAC$. Enunciate the converse propositions and say whether they are true.

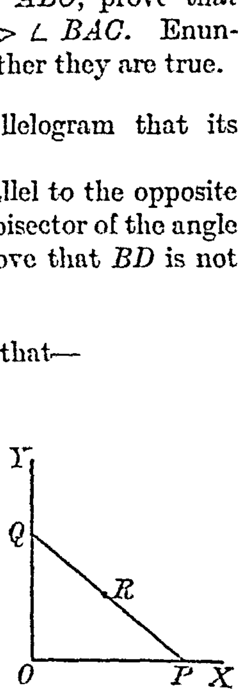
2. Prove from the definition of a parallelogram that its diagonals bisect one another.

Through the point $C$ a line is drawn parallel to the opposite side $AB$ of a triangle $ABC$, and meeting the bisector of the angle $BAC$ at $D$. If $AB$ is not equal to $AC$, prove that $BD$ is not parallel to $AC$.

3. Show, by drawing geometrical figures, that—

(i) $(x+5)(x+3) = x^2 + 8x + 15$.

(ii) $(x-5)(x+3) = x^2 - 2x - 15$.




4. A straight rod $PQ$ slides so that $P$ moves along $OX$ and $Q$ along $OY$; draw the locus traced out by the middle point $R$, taking twice the dimensions of the figure.

5. If two chords $AB$, $CD$ of a circle are parallel, prove that $AC = BD$.

Two circles touch externally at $P$ and a line cuts one of the circles at $A$, $B$, and the other at $C$, $D$; $AP$ and $BP$ are produced to meet the second circle at $E$ and $F$ respectively. Prove that $EF$ is parallel to $AD$ and that the angle $EPC$ is equal to the angle $BPD$.

6. Construct a triangle $ABC$ in which $AD$, the perpendicular from $A$ to $BC$, is 3 in., $AG$ ($G$ being the intersection of the medians) is 2·4 in., and the radius of the circumcircle is 2 in.

## No 44

1 An equilateral triangle $CDE$ is described on the side $CD$ of a square $ABCD$, the diagonals cut $DE$, $CE$ at $F$ and $G$ respectively Prove that $EF = EG$

2 State the various ways you know of proving that two lines are parallel

In the figure of Ques 1 prove that $FG$ is parallel to $DC$

3 State how to draw a line the length of which is exactly $\sqrt{21}$ in and prove your construction to be correct

4 Prove that the greatest line that can be drawn from a point outside a circle to a point on the circumference is that which passes through the centre

Two points $A$ and $B$ are fixed and are outside a given circle Find a point $P$ on the circle so that $PA^2 + PB^2$ has the greatest possible value

5 Circles are inscribed in each of the four triangles into which a parallelogram is divided by its diagonals Prove that the quadrilateral formed by joining the centres of these circles is a rhombus

6 Unequal perpendiculars $AP$, $BQ$ are drawn at the extremities $A$ and $B$ of a straight line $AB$ and a circle is described on $PQ$ as diameter cutting $BQ$ the larger of the two perpendiculars at $R$ Prove that $AP = BR$

If this circle cuts $AB$ at $X$ and $Y$ prove that $AX = BY$, and if $AP = 1$ $AB = a$ and $BQ = b$ prove that the lengths of $AX$ and $AY$ are the roots of the equation $x^2 - ax + b = 0$

## No. 45

1. Prove that the middle point of any straight line which meets two parallel straight lines is equidistant from these lines.

Explain how to find, without measuring any line or angle, a point which is equidistant from each pair of two pairs of parallel straight lines.

2. In a quadrilateral $ABCD$ the diagonal $BD$ bisects the quadrilateral and is of length 5 cm.; $AB = 3{\cdot}9$ cm., $CD = 3$ cm., and angle $BAD = 115°$. Construct the quadrilateral and measure $BC$.

3. Prove that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

From the vertices $B$, $C$ of a triangle $ABC$ perpendiculars $BX$, $CY$ are let fall on the opposite sides. Prove that—

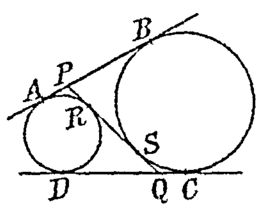
(i) $AX \,.\, AC = AY \,.\, AB$;

(ii) $BY \,.\, BA + CX \,.\, CA = BC^2$.

4. Draw two non-parallel straight lines which do not meet on your paper. Explain how to draw a line which would, if produced, bisect the angle between the original two lines.

5. Show how four common tangents may be drawn to two non-intersecting circles.

The figure shows two circles with two direct common tangents and one transverse. Prove $AB = PQ$.




6. $AB$ is the diameter of a circle. Through $A$ two chords $AP$, $AQ$ are drawn and produced to meet the tangent at $B$ in $R$ and $S$ respectively. Prove that, whether $AP$, $AQ$ are on the same side of $AB$ or on opposite sides, the four points $P$, $Q$, $R$, $S$ are concyclic.

4—41.1193

## No 46

1 $OX$, $OY$ are two mirrors, the reflecting faces containing an angle of 40° $P$ is a luminous point 3 ft from $OX$ and 2 ft from $OY$ Draw a diagram on the scale 1 cm = 1 ft showing $P$ and four images $P_1$ $P_2$ $P_3$ $P_4$ Prove that the points $P, P_1$ $P_2$ $P_3$ $P_4$ lie on a circle

2 If a parallelogram and a triangle are on the same base and between the same parallels prove that the area of the parallelogram is double that of the triangle

$ABCD$ is a parallelogram in which $AB$ is greater than $AD$ A point $H$ is taken inside the triangle $ACD$ Prove that—

$$\triangle AHC = \triangle AHB - \triangle AHD$$

3 Draw figures to illustrate the algebraical formula—

(i) $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$

(ii) $(a+b)^2 - (a-b)^2 = 4ab$

4 In any circle prove that equal chords are the same distance from the centre

Draw a circle radius 2 in and take a point $P$ 3 2 in from the centre Show with proof how to draw a line $PAB$ cutting the circle at $A$ and $B$ so that $AB = 2\ 9$ in

5 If at a point on a circle a tangent and chord are drawn either angle between the tangent and chord is equal to the angle in the segment on the other side of the chord

From a fixed point $O$ tangent $OA$ $OB$ are drawn to a fixed circle any secant $OPQ$ is drawn and a chord $BR$ parallel to the secant If $AR$ meets $PQ$ at $X$ prove that $X$ lies on a fixed circle Hence show that $X$ is the mid point of $PQ$

6 $AB$ is a chord of a circle and $C$ is any point outside the circle Show with proof, how to draw a secant $CDE$ so that $DE$ is bisected by $AB$

## No. 47

1. Prove that two quadrilaterals $ABCD$, $PQRS$ are congruent if $AB = PQ$, $BC = QR$, $CD = RS$, and $\angle ABC = \angle PQR$, $\angle BCD = \angle QRS$.

2. Prove that the angles at the base of an isosceles triangle are equal.

In a triangle $ABC$ the sides $AB$, $AC$ are equal and the base $BC$ is produced through $C$ to any point $D$. From $D$, $DE$ is let fall perpendicular to $AB$ and $DF$ perpendicular to $AC$ produced. Prove that $BD$ bisects the angle $EDF$.

3. If a number of parallel lines divide any transversal into equal parts, prove that any other transversal will also be divided into equal parts.

Draw a line $AB$ of length 3·7 in., divide it into two parts $AP$, $BP$ such that $3AP = 4BP$. Give a proof.

4. Explain how to divide a straight line $AB$ into two parts at $C$ so that rectangle $AB \,.\, BC$ is equal to the square on $AC$.

If $ABDE$ is the rectangle $AB \,.\, BC$, prove that $AD^2 = 3\ AC^2$.

5. Prove that the line from the centre at right angles to a chord bisects the chord.

Two circles intersect at $A$; show, with proof, how to draw a line $PAQ$ meeting the circles at $PQ$ so that $PQ$ is bisected at $A$.

6. Explain how a regular pentagon may be drawn without using a protractor.

## No. 48

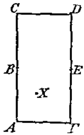
1. Two equilateral triangles $BAC$, $QAP$ have a common vertex $A$; show that the sides of the triangle $QPC$ are respectively equal to $QA$, $QB$, $QC$.

2. What is the locus of points equidistant from the arms of a given angle?

A point $Q$ is taken in the base $YZ$ produced of a triangle $XYZ$ in which $XY = XZ$; perpendiculars are let fall from $Q$ on $XY$, $XZ$, produced if necessary. Show that the difference between the perpendicular is the same whatever point $Q$ is taken in the produced base.

3. Construct a triangle $ABC$, being given that median $AD = 4.6$ cm, median $BE = 3.5$ cm, and side $AB = 4$ cm.

4. The figure represents a billiards table 12 ft $\times$ 6 ft, with pockets at $A, B, C, D, E, F$; a ball is placed at $X$, 2 ft from $AC$ and 3 ft from $AF$. Regarding the ball and pockets as points, and assuming that the paths of the ball before and after striking a side are equally inclined to the side draw a scale figure to find (i) a point $P$ in $AC$ such that the ball after striking at $P$ may go into the pocket $D$, and (ii) a point $Q$ in $AC$ such that the ball after striking at $Q$ may hit the top cushion $CD$ and then go into the pocket $F$.




5. The sides $BC$, $CA$, $AB$ of a triangle $ABC$ are respectively 6 in., 9 in., 10 in. long. The inscribed circle of the triangle touches these sides at $D, E, F$ respectively. Calculate the lengths of $BD$, $CE$, $AF$. If the inscribed circle touching $BC$, not produced, touches $AB$, produced at $P$, calculate the length of $AP$.

6. From a point $P$ on the circumcircle of a triangle $ABC$ perpendiculars $PH$, $PK$, $PL$ are let fall on the sides $BC$, $CA$, $AB$, produced if necessary. Prove that $H$, $K$, $L$ lie on a straight line.

## No. 49

1. A pirate buried some treasure at a place $C$, near two trees $A$ and $B$, $B$ being 57 yds. W. of $A$. He made a note that the angle $CAB$ was $50^\circ$ N. of $AB$ and $BC$ was equal to 47 yds. On returning some years later he made correct measurements, but failed to find the treasure. After consideration he made a second attempt and found the treasure. Draw a figure to the scale 1 cm. = 10 yds. to explain his mistake. What was the distance between the two holes ?

2. The middle point $E$ of the side $AD$ of a parallelogram is joined to $B$, and $BE$ cuts the diagonal $AC$ at $F$. Prove that the triangle $BCF$ is one-third of the parallelogram.

3. Four points $A, B, C, D$ are taken in this order on a straight line ; prove (i) by means of a figure, (ii) algebraically, that $AD^2 + BC^2 = AC^2 + BD^2 + 2\,AB\,.\,CD$.

4. Prove that the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals together with four times the square on the line joining the middle points of the diagonals.

5. Show that the acute angle between a tangent and a chord through the point of contact is half the angle at the centre subtended by the chord.

Two circles touch internally at $A$ and a chord $ABC$ meets the circles at $B$ and $C$. The tangent at $B$ meets the outer circle at $D$ and $E$ ; prove that the tangents at $B$ and $C$ are parallel and that the arc $CD$ is equal to the arc $CE$.

✓6. Construct a triangle $ABC$ in which the radius of the inscribed circle is 1 in., the distance of $A$ from the centre of that circle is 2·5 in., the side $BC$ is 4 in.

## No 50

1 A point $P$ is taken inside a triangle $ABC$ prove that $AP + BP + CP$ is less than the sum of the sides of the triangle $ABC$ but greater than half the sum of the sides

2 Prove that the bisector of the angles of any quadrilateral enclose a cyclic quadrilateral

If this cyclic quadrilateral is a rectangle, show that the original quadrilateral must be a parallelogram

3 State and prove a rule for finding the area of a trapezium

Draw a trapezium $ABCD$ in which $AB$ is parallel to $CD$ and in which $AB = 3$ 4 cm , $BC = 4$ cm , $CD = 6$ cm , $DA = 4$ 3 cm Use your rule to find the area of the trapezium

4 Through a point $P$ 3 cm from the centre $O$ of a circle with radius 5 cm chords are drawn Show that the locus of the middle points of these chords is a circle Draw the locus State with proof, how to draw the longest and shortest of these chords and calculate their lengths

5 Enunciate the proposition concerning the tangent and a secant drawn to a circle from an external point

Explain, with proof, how to draw a circle to pass through two given points and to touch a given circle

6 A triangle $ABC$ being given, explain how to draw three circles with their centres at $A$, $B$, and $C$, such that each circle touches the other two

If these circles touch at $D$ $E$, and $F$ and if the $\angle D = 30°$ and $\angle E = 70°$ calculate the angles of the triangle $ABC$

## No. 51

1. Equilateral triangles $PBC$, $QCA$, $RAB$ are described on the sides of a triangle $ABC$, all falling outside the triangle. Prove that $PA = QB = RC$.

2. In any triangle prove that the sum of the squares on two sides is equal to twice the square on half the third side, together with twice the square on the median bisecting the third side.

Prove that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on the four sides.

3. Prove that an angle at the centre of a circle is double any angle at the circumference standing on the same arc.

$AB$, $AC$ are equal chords of a circle with centre $O$, and $AD$ is another chord cutting $BC$ at $E$. Prove that the angle $AEC$ is one half of one of the angles $AOD$.

4. State the construction for inscribing a circle in a triangle and prove it to be correct.

Find, by drawing, the radius of the biggest sphere that can be covered by a cone of height, 8 in., and with a base of radius of 6 in.

5. Prove that a straight line parallel to one side of a triangle divides the other two sides in the same ratio.

A point $P$ is taken on the side $AB$ of a triangle $ABC$, such that $AP = 2PB$. Through $P$, $PQ$ is drawn parallel to $BC$, meeting $AC$ at $Q$, and $PR$ is drawn parallel to $BQ$, meeting $AC$ at $R$. Find the ratio $AR : RC$.

6. In a triangle $ABC$, the angle $C = 90°$ and the angle $B = 30°$; $CB$ is produced to $D$ so that $BD = BA$. From this figure calculate tan 15°, sin 15°, each to three decimal places.

## No. 52

1 Define a parallelogram, and prove that if both pairs of opposite sides of a quadrilateral are equal the quadrilateral is a parallelogram

$P, Q, R$ $S$, are the middle points of the sides of an irregular quadrilateral, prove that $PR$, $QS$ bisect one another

2 *Prove that all points equidistant from two fixed points* $A$ and $B$ lie on a certain fixed straight line

Draw a triangle $ABC$ having $AB = 6\ 7$ cm, $BC = 7\ 8$ cm $CA = 8\ 9$ cm, on $AB$ construct a triangle $VAB$, such that $VA = VB = VC$

3 On two sides, $AB$, $AC$ of a triangle as diameters circles are described Prove that $D$, the other point of intersection lies on $BC$ or $BC$ produced

4 Explain what is meant by *similar figures* If two triangles have their corresponding sides in the same ratio, prove that they are equiangular

5 If four lines are in proportion give a geometrical proof that the rectangle contained by the means is equal to the rectangle contained by the extremes

From a point $P$, two tangents $PA$, $PB$ are drawn to a circle with centre $O$, and $PO$ cuts $AB$ at $C$ Prove that $PC.PO = PA^2$

6 A point $P$ is taken distant $a$ inches from the centre $O$ of a circle with radius $r$, from a point $Q$ on the circumference $QN$ is let fall perpendicular to $PO$, produced if necessary If the angle $QON$ is $x°$, find expressions for the lengths of $PN$, $QN$, $PQ$

From the expression for $PQ$ deduce that $PQ$ is greatest or least when it lies along the line $PO$

## No. 53

1. Two isosceles triangles $PAB$, $QAB$ are on the same side of the same base (i.e. unequal side) $AB$. Prove that $PQ$ when produced bisects $AB$.

2. If two triangles have two sides of one equal respectively to two sides of the other, but the angle contained by one pair greater than the angle contained by the other pair, then the third side of the triangle with the greater included angle is greater than the third side of the other.

In a quadrilateral $ABCD$ the opposite sides $AD$, $BC$ are equal, but the angle $DAB$ is greater than the angle $ABC$. Prove that the angle $BCD$ is greater than the angle $ADC$.

3. Prove that a triangle is right-angled if the square on one side is equal to the sum of the squares on the other two.

Calculate the area of a quadrilateral $ABCD$ in which $AB = 6$ cm., $BC = 2{\cdot}5$ cm., $CD = 3{\cdot}3$ cm., $DA = 5{\cdot}6$ cm., angle $ABC = 90°$.

4. Prove that the two tangents drawn to a circle from an external point are equal.

A quadrilateral is such that the sum of one pair of opposite sides is equal to the sum of the other pair of opposite sides. Prove that the bisectors of the angles of the quadrilateral are concurrent.

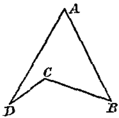
5. Prove that the bisector of an angle of a triangle divides the opposite side proportionally to the sides containing the angle.

The middle points of the sides $BC$, $CA$, $AB$ of a triangle are $P$, $Q$, $R$ respectively, and $X$ is a point in $QR$ such that $QX : XR = AB : AC$. Prove that $PX$ bisects the angle $RPQ$.

6. From a point $P$ a tangent $PT$ is drawn to a circle of which $TR$ is a diameter, and $PR$ cuts the circle at $Q$. Calculate the length of $PQ$, being given $TR = 9{\cdot}6$ and angle $TPQ$ is $63° \, 15'$.

## No. 54

1 It is required to make an exact copy of the quadrilateral $ABCD$ by measuring one length and as many angles as are *necessary* Record your measurements of the length (to the nearest tenth of an inch) and angles, and explain why they are sufficient to make a copy of the figure




2 If in a triangle one side is greater than another, prove that the angle opposite the greater side is greater than the angle opposite the less

Construct a triangle $ABC$ so that $BC = 3{\cdot}4$ in, angle $B = 42°$, and $AB$ exceeds $AC$ by $1{\cdot}6$ in Measure $AC$

3 Construct, without any calculation a rectangle with one side $1{\cdot}7$ in equal in area to a rectangle with sides $3{\cdot}2$ in and $1{\cdot}2$ in State and prove your construction

4 If two chords of a circle intersect internally or externally, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other

Circles are drawn to pass through two fixed points $A$ and $B$ What are the respective loci of—

(i) The centres of these circles ?

(ii) The points of contact of the tangents drawn to these circles from a fixed point $C$ in $AB$ produced ?

5 Explain how to draw a figure similar to one rectilineal figure $A$, and equal to another rectilineal figure $B$

Construct geometrically an isosceles triangle whose sides are in the ratio $3 : 3 : 2$ equal in area to a square of side $2{\cdot}5$ in

6 (i) Construct, without using tables, an angle $A$, being given $\sin \frac{A}{2} = {\cdot}8$, and from the figure find the values of $\sin A$ and $\cos A$

Check your result by using protractor and tables

(ii) State and prove a formula giving the cosine of an angle of a triangle which is not right angled in terms of the sides

## No. 55

1. State and prove a formula for obtaining the size of an internal angle of a regular polygon of $n$ sides.

A square $ABCD$ and a regular pentagon $ABPQR$ are described on opposite sides of a line $CD$. Find, by calculation, the angles of the pentagon $PQRDC$.

2. Prove that the opposite sides of a parallelogram are equal.

In a parallelogram $ABCD$ the side $AB$ is greater than the side $AD$. Prove that the point $E$, where the bisector of the angle $A$ cuts the side $DC$, is between $C$ and $D$.

3. Prove (using, if you like, the properties of similar triangles) that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

$AB$ and $PQ$ are chords in two circles with centres $C$ and $O$ respectively, the former circle having the longer radius; and $AB$, $PQ$ are at equal distances from $C$ and $O$ respectively. Prove that $AB$ is greater than $PQ$.

4. $ABCD$ is a cyclic quadrilateral in which $AD$ is equal to $CD$; $I$ is the centre of the inscribed circle of the triangle $ABC$. Prove that $D$ is the centre of the circumcircle of the triangle $AIC$.

5. The bisector of the angle $A$ of a triangle $ABC$ meets $BC$ at $D$; $I$ is the centre of the inscribed circle and $E$ of the described circle touching $BC$, not produced. Prove that the rectangle $BI \,.\, DE$ is equal to the rectangle $CE \,.\, DB$.

6. The ground at the foot $A$ of a vertical wall $AB$ slopes downwards at an angle 32° with the horizontal. A ladder 18 ft. long, placed at a point $C$ on the ground, just reaches $B$, the top of the wall. If $AC$ is equal 5 ft. 8 in., calculate the height of the wall and the inclination of the ladder to the vertical.

## No. 56

1 If two straight lines $AB$ $CD$ intersect at $O$, and the opposite vertical angles $AOC$, $BOD$ are bisected by $OP$, $OQ$ respectively, prove that $PO$ and $OQ$ are the same straight line

If $X$, a point on the bisector of the angle $AOD$, is joined to $Y$, any point in $OP$, prove that the circle having $XY$ as diameter will pass through $O$

2 On the same side of a line $XO$, draw angle $XOC$ equal to $50°$, and another angle $XOY$ equal to $117°$, and make $OC$ $= 1\ 7$ in Draw a straight line through $C$, cutting $OX$ at $A$ and $OY$ at $B$, such that $AC = CB$ State and prove your construction

3 Prove that the opposite angles of a cyclic quadrilateral are equal to two right angles (Try to prove this without using any property of a circle, except that the radii are equal)

The opposite sides $AB$, $DC$ of a quadrilateral $ABCD$ meet at $P$ when produced and the rectangle $PA$ $PB$ is equal to the rectangle $PC$ $PD$ Prove that the angles $DAC$, $DBC$ are equal

4 The sides of a triangle $ABC$ are $a$ $b$ $c$, show that the radius of the inscribed circle is equal to twice the area of the triangle divided by $(a+b+c)$

Calculate the length of the radius of the inscribed circle of a triangle whose sides are 73 cm 55 cm 48 cm

5 Prove that two triangles are similar if they have two sides of the one proportional to two sides of the other, and the angles included by those sides equal

Tangents $PA$, $PB$ are drawn to a circle with centre $O$, and $PO$ cuts $AB$ at $C$, $XY$ is any other chord through $C$ Prove that—

(i) Rectangle $PC$ $CO = CA^2$

(ii) Triangles $PCX$, $OCY$ are similar

(iii) $P$, $X$, $O$, $Y$ lie on a circle

(iv) $PO$ bisects the angle $XPY$

6 A road runs east from $A$ and on the north side of the road is a forest A man walks 6 miles from $A$ along the road, and then turns northward through an angle of $70°$, and walks 5 miles into the forest Find by calculation, how far he is then from $A$ and from the road In what direction must he walk to return direct to $A$ ?

## No. 57

1. What do you know about the diagonals of (i) a parallelogram, (ii) a rhombus, (iii) a rectangle, (iv) a square, (v) a cyclic quadrilateral.

In a quadrilateral $ABCD$ the diagonals $AC$ and $BD$ are equal, and the angles at $B$ and $C$ are both right angles. Prove that the quadrilateral is a rectangle.

2. The diagonals $AC$, $BD$ of a quadrilateral are at right angles and intersect at $O$; show by a figure that the area of the quadrilateral is half the rectangle to $AC$, $BD$.

3. Through a point $P$ in the diagonal $AC$ of a parallelogram $ABCD$ are two lines $HPK$, $XPY$, each parallel to two sides of the parallelogram, meeting $AB$ at $Y$, $BC$ at $K$, $CD$ at $X$, $DA$ at $H$. Prove that—

(i) The parallelograms $DHPX$, $BKPY$ are equal in area;

(ii) The parallelograms $AYPH$, $CXPK$ are similar to one another and to $ABCD$.

4. Draw, and explain, figures to illustrate the algebraical formula—

(i) $(a - b)^2 = a^2 - 2\,ab + b^2$;

(ii) $(a + b)^2 - (a - b)^2 = 4\,ab$.

5. A quadrilateral $ABCD$ is such that a circle can be drawn to touch all four sides; prove that—

$$AB + CD = AD + BC.$$

If $P$, $Q$, $R$, $S$ are the respective points of contact with the sides $AB$, $BC$, $CD$, $DA$, express the angles of the quadrilateral $PQRS$ in terms of the angles of the quadrilateral $ABCD$.

6. A quadrilateral $ABCD$ is determined by the measurements $AB = 3$ in., $BC = 3{\cdot}8$ in., $CD = 2{\cdot}9$ in., $BAD = 106°$, $BCD = 90°$.

Construct the quadrilateral; find its area by constructing a triangle equal to it in area.

What are the corresponding measurements of a similar quadrilateral $A'\,B'\,C'\,D'$ of which the area is $\frac{4}{9}$ the area of $ABCD$?

## No. 58

1 Prove that, if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, the two triangles are equal in every respect

$P$ and $Q$ are two points on the same side of a straight line $XY$, but at unequal distances Find a point $O$ in $XY$ such that $PO$, $QO$ may be equally inclined to $XY$

2 Prove that in any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the third side and the projection on it of the median bisecting that side

3 Construct a parallelogram $ABCD$ of area 3 sq in, such that $AB = DC = 1\ 5$ in and the perpendicular distance between $AD$ and $BC$ is 1 25 in

Explain your construction

4 Define a tangent to a circle Use your definition to prove that the tangent to a circle at any point on it is perpendicular to the radius drawn to that point

Two circles touch at $A$, from $P$ any point on the common tangent at $A$, tangents $PQ$, $PR$ are drawn to the circles Show, with proof, how to draw a circle to touch the two circles at $Q$ and $R$ respectively

5 State the ratio property of the bisector of an angle of a triangle

Two points $A$ and $B$ are 3 cm apart, draw the locus of a point $P$ which moves so that $PA : PB = 2 : 1$

6 A man, whose eyes are 5 ft 6 in above the ground, stands at a horizontal distance 30 ft from the foot of a vertical tower 55 ft high Find, as nearly as your tables permit, the angle subtended by the tower at the man's eye

## No. 59

1. Define a parallelogram ; from your definition prove that either diagonal bisects the parallelogram.

Any point $E$ is taken on the diagonal $AC$ of a parallelogram $ABCD$ ; prove that the triangles $BCE$, $DCE$ are equal in area.

2. Prove that triangles on the same base and between the same parallels are equal in area.

Prove that of all such triangles the isosceles triangle has the smallest perimeter.

3. If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts is equal to the difference of the squares on half the line and the line between the points of section.

Explain how to divide a given finite straight line so that the rectangle contained by the parts may have (i) the maximum area, (ii) the maximum perimeter.

4. Of two chords of a circle drawn through a given point, prove that the one farther from the centre is less than the other.

A point being given within a given circle, explain how to draw a chord so that the rectangle contained by the segments shall have (i) the minimum area, (ii) the minimum perimeter.

5. Divide, by a geometrical construction, a line $AB$, 3 in. long, at a point $P$, so that $AP^2 = 2\,BP^2$.

Do the same by algebra and thus check your result.

6. If $A$ is an acute angle of a triangle with hypotenuse $c$, show that the area is $\frac{1}{2}\,c^2 \sin A \cos A$. Deduce the maximum area for a right-angled triangle with hypotenuse $c$ and verify by pure geometry.

## No. 60

1 Without using a protractor, construct an angle of $52\frac{1}{2}°$ Give proof that your construction is correct

2 If two sides of a triangle are equal, prove that two of the angles are equal

On the equal sides $AB$, $AC$ of a triangle, squares $ACDE$, $ABHK$ are described Prove that $KE$ is parallel to $BC$.

3 Explain how to make a square equal to a given rectangle

In a straight line $AB$ 3 5 in long, find, by a geometrical construction, a point $P$ such that $AP^2 - PB^2$ may equal the square on a line 2 in long

4 Prove that the perpendicular from the centre of a circle to any chord bisects the chord

From the extremity $A$ of a diameter $AB$ of a circle, chords are drawn What is the locus of the middle points of these chords ?

5 Two circles touch one another and also touch a straight line at points $A$ and $B$ Prove that $AB$ is a mean proportional between the diameters of the circles

6 A rod $OP$ of length $a$ inches is turned about $O$ by another rod $AP$ of length $b$ inches of which the end $A$ is compelled to move to and fro along a portion of the line $OX$ Show that, when the angle $AOP$ is $x°$, $OA = a \cos x + \sqrt{b^2 - a^2 \sin^2 x}$ If $a = 7$, $b = 12$, find the length of that part of $OX$ along which $A$ moves

Also find the greatest value of the angle $OAP$

## No. 61

1. Prove, without using proportion, that the straight line drawn through the middle point of one side parallel to another side bisects the third side.

Prove that the line joining $E$ and $F$, the middle points respectively of $AD$ and $BC$, the non-parallel sides of a trapezium, is parallel to $AB$ and $CD$, and equal to half their sum.

2. Prove that the three medians of a triangle are concurrent.

Construct a triangle, being given that the lengths of the three medians are 2·7, 3·4, 1·9 respectively.

3. Prove that in equal circles equal chords subtend equal angles at points in the circumference.

Two equal circles cut at $A$ and $B$; with $A$ as centre a circle is described cutting the two circles at $C$ and $D$ on the same side of $AB$. Prove that $BCD$ is a straight line.

4. The inscribed circle of a triangle $ABC$ touches the sides $CA$, $AB$, at $E$, $F$ respectively. Prove that $AE$ or $AF$ equals half the perimeter of the triangle diminished by the side $BC$.

Draw a triangle $ABC$ in which $BC = 7·3$ cm., $CA = 6·9$ cm., $AB = 5·1$ cm. Now with centres $A$, $B$, and $C$ draw three circles, each of which touches the other two.

5. If a perpendicular be let fall from the right angle of a right-angled triangle upon the hypotenuse prove that the triangle is divided into triangles similar to the whole triangle and to one another.

From a point $P$ a tangent $PD$ is drawn to a given circle; from $D$ a perpendicular $DC$ is drawn to the diameter $AB$, which, when produced, passes through $P$. Prove that $AC : CB = AP : PB$.

6. In a triangle $ABC$, $AB = 13$ in., $AC = 4$ in., and $AD$, the perpendicular on $BC$, $= 3·2$ in. Calculate the magnitudes of the angles $ABC$ and $ACB$ and the length of $BC$.

## No. 62

1 Prove that the diagonals of an equal-sided quadrilateral bisect one another at right angles

$ABCD$ is a square, explain, with proof, how to draw a line parallel to the diagonal $BD$ and cutting $AB$ at $P$, and $AD$ at $Q$, such that $PQ = AB$

2 Construct a quadrilateral $ABCD$, having $AB = 4$ in, $BC = 2$ in, $CD = 3\ 5$ in, angle $BAD = 60°$, angle $ABC = 110°$, and the angle $ADC$ acute Bisect the straight quadrilateral by a straight line drawn through $B$ State and prove your con struction

3 Give conditions that four points may be concyclic

If four circles are drawn so that each circle touches two only of the other three, determine whether the four points of contact are concyclic

4 The perpendiculars $AD$, $BE$, $CF$, drawn from the vertices of a triangle to the opposite sides intersect at $K$ and meet the opposite sides at $D$ $E$, $F$ Prove that $AK\ \ KD = BK\ \ KE = CK\ \ KF$

5 If in a triangle $ABC$ the external bisector of the angle $A$ meets $BC$, produced at $D$, prove that $BA\ \ AC = BD\ \ DC$

Two points $A$ and $B$ are at a distance 4 cm Construct the locus of a point which moves so that $AP\ \ PB = 2\ \ 3$

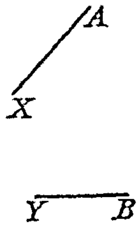
6 A path $AX$ is drawn from the foot $A$ of a column $AB$, 75 ft high, on the top of which is a statue $BC$, 16 ft high State how to find the point $P$ in $AX$ such that the angle $BPC$ is as large as possible Find the size of that maximum angle by trigonometrical calculation

## No. 63

1. At the extremity $A$ of a line $AB$, which cannot be produced through $A$, erect, without using protractor or set square, a line $AC$ perpendicular to $AB$. Prove your construction.

2. Take two points $A$ and $B$ in the left-hand edge of your paper and draw two diverging lines $AX$ and $BY$. Construct the line $CZ$, which would, if produced off the paper, bisect the angle between $XA$ and $YB$ if they also were produced off the paper to meet. State and prove your construction.

3. In the Figure, $A$ and $B$ are the opposite corners of a parallelogram $PAQB$, but $AX$ and $BY$ cannot be produced to meet; if they could, $P$ would be their point of intersection. Construct as much as you can of the diagonal $PQ$. State and prove your construction.




4. In the Figure of Question 3, suppose that $AX$ and $BY$ are obtained by producing two adjacent sides of a parallelogram, the lengths of the sides being 2 in. and 3 in. respectively. Construct the line $PQ$, which is the prolongation of the diagonal through the intersection of those sides. State and prove your construction.

5. Draw an arc $AB$ of a circle. Without finding the centre draw the tangents at $A$ and $B$.

Take any point $P$ outside the segment but within the triangle formed by the chord $AB$ and the tangents at $A$ and $B$; show how the tangent from $P$ may be drawn without finding the centre.

6. A sphere of radius 6 in. rests on a hollow cylinder of radius 4 in.; find by drawing and by calculation, the distance of the centre of the sphere from the centre of the top rim of the cylinder.

## No 64

1 Prove that if two angles of a triangle are equal two sides also are equal.

A point $P$ in the hypotenuse $AB$ of a right angled triangle $ABC$ is such that the angle $PAC$ = angle $PCA$ prove that $AP = PB$

2 $AB$ is a line 3 5 cm long draw the complete locus of a point $P$ which moves so that the area of the triangle $PAB$ is 3 5 sq cm and that no side of the triangle is greater than $AB$

3 Prove that the sum of the squares on two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median bisecting the third side

In a quadrilateral $ABCD$ $E$ is the middle point of the diagonal $AC$ and $F$ of $BD$ Prove that the sum of the squares on the sides of the quadrilateral is less than the sum of the squares on the diagonals by four times the square on $EF$

4 If the opposite sides of a quadrilateral are together equal to half the perimeter of the quadrilateral prove that a circle can be described to touch all the sides of the quadrilateral

5 Explain how to draw a rectilineal figure similar to a given rectilineal figure and with its area $p$ times the area of the given figure

As an example draw an equilateral triangle five times as large as the equilateral triangle with side 1 3 in

6 A cone of height 7 cm with vertex $V$ has a circular base whose diameter $AB$ is 4 7 cm Draw to scale the section of the cone made by a plane parallel to the base the area of the section being one third the area of the base

Find by calculation the size of the angle $AVB$

## No. 65

1. Give the enunciations of three propositions which state that a straight line is greater than some other line.

Draw a triangle $ACB$ having $AB = 2\cdot3$ in., $AC = 1\cdot7$ in., $BC = 1\cdot1$ in. Now show within what area a point $P$ must fall in order that $AP$ may be greater than 1·7 in., but $BP$ less than 1·1 in.

2. Lines are drawn through the vertices of a triangle $ABC$ parallel to the opposite sides, forming the triangle $XYZ$. Show that the area of $XYZ$ is four times that of $ABC$.

Prove the medians of any triangle are also the medians of the triangle formed by joining the middle points of the sides.

3. Construct a parallelogram with one side $AB = 3\cdot2$ cm., the angle $ABC = 50°$, and the diagonal $BD = 5\cdot8$ cm.

4. Take any four points $A$, $B$, $C$, $D$ in a straight line and show how to find a point $O$ in the line $AD$ so that the rectangle $OA \,.\, OC =$ rectangle $OB \,.\, OD$.

5. Prove that two similar rectilineal figures can be divided into pairs of similar triangles.

$ABC$ is an equilateral triangle with side 2 in., $XYZ$ is an equilateral triangle with side 1·7 in.; a point $P$ is taken inside the triangle $ABC$ so that $AP = 1\cdot3$ in. and $PC$ bisects the angle $C$. Find a point $O$ in the triangle $XYZ$ such that it will divide $XYZ$ into triangles similar to $PAB$, $PBC$, $PCA$.

6. A spherical balloon of radius 20 ft. subtends an angle 10° 18′ at an observer's eye when the angle of elevation of the centre is 56° 41′. What is the height of the centre above the horizontal level of the observer's eye?

## No 66

1 State and prove the congruency enunciation which does not mention any angles

A triangle $ABC$ is formed by three thin rods $AB$, $BC$, $CA$, the triangle is turned about $B$ through an angle to the position $A'\,BC'$ What does the above mentioned enunciation tell you about the angles of the triangle $A'\,BC'$? Prove that one of the angles between $AC$ and $A'\,C'$ is equal to the angle $ABA'$

2 Through the middle point $D$ of the side $BC$ of a triangle $ABC$ a line is drawn making equal angles $AXD$, $AYD$, with $AB$, $AC$ respectively Prove that $AX + AY = AB + AC$

Draw any angle $PAQ$ and take a point $D$ within the angle $QAP$ Show how to construct a triangle $ABC$ so that $BC$ passes through $D$, meets $AP$, $AQ$ at $B$ and $C$ respectively, and so that the triangle $ABC$ has the smallest possible area

3 Define a tangent to a circle and from the definition prove that the perpendicular to any radius at its extremity is a tangent to the circle

What other properties of a tangent do you know?

What is the locus of points from which the tangents to given two intersecting circles are equal?

4 Draw a circle with radius 75 in about it describe, without using a protractor, a triangle having two of its angles 30° and 45° respectively

5 $AD$ and $PS$ are medians of two equiangular triangles $ABC$, $PQR$ Prove that $AD \quad PS = BC \quad QR$

6 The circumcircle of an isosceles triangle $ABC$ meets $AD$, the perpendicular bisector of $BC$, when produced at $E$ Through $D$ any chord $PDQ$ is drawn, and $EP$, $EQ$ cut $BC$, produced when necessary at $R$ and $S$ Prove that the rectangle $DR \quad DS$ is constant

## No. 67

1. Prove that the opposite sides and angles of a parallelogram are equal.

$ABCD$ is a parallelogram; $AB$ is produced to a point $E$ such that $DE$ bisects $BC$. Prove that $AB = BE$.

2. Divide a straight line 3·8 in. long into two parts so that the rectangle contained by them may be of area 3 sq. in. State and prove your construction.

3. Prove that the perpendicular bisector of any chord of a circle must pass through the centre.

Perpendicular $AD$, $BE$ are let fall from the vertices $A$ and $B$ of a triangle $ABC$ to meet the opposite sides at $D$ and $E$; $K$ is the point of intersection of $AD$ and $BE$. Prove that the line joining the mid-points of $AB$ and $CK$ bisects $DE$ at right angles.

4. Given a square whose side is 1·5 in., describe a regular octagon by cutting off the four corners. Explain your construction.

5. A straight $AB$ is divided internally at $P$ and externally at $Q$ in the same ratio and $O$ is the middle point of $AB$.

Prove (i) $OP, OQ = OA^2$, (ii) $\dfrac{1}{AP} + \dfrac{1}{AQ} = \dfrac{2}{AB}$

6. Give a geometrical construction, with proof, for making an isosceles triangle with each of the base angles double of the vertical angle.

Deduce that $\sin 54° = \dfrac{\sqrt{5}+1}{4}$.

## No 68

1 Prove that the sum of the interior angles of a polygon of $n$ sides is equal to 180 $(n-2)$ degrees

In a pentagon $ABCDE$ the angles at $A$ and $B$ are each 110° and the angles at $C$ and $E$ are each 100° The alternate sides are produced to meet, forming a star shaped figure What is the sum of all its angles ? Calculate the magnitudes of the angles and verify that the sum is right

2 Show that if one diagonal of a quadrilateral divides it into two equal parts it need not be a parallelogram , but if both diagonals divide it into equal parts it must be a parallelogram

3 If two circles touch, prove that the centres and point of contact are in the same straight line

A circle is drawn touching a given circle whose centre is $A$ and also a given straight line $BC$ Show that the centre is equidistant from $A$ and from one of two straight lines parallel to $BC$

4 Prove geometrically that, if a straight line $AB$ be divided equally at $O$ and unequally at $X$,

$$AX \cdot XB = AO^2 - OX^2$$

Divide a straight line 7 cm long into two parts so that the difference of the squares on these parts may be 6 sq cm

5 Without assuming any propositions about a circle prove that the angle in a semicircle is a right angle

Two points $P$ and $Q$ 1 in apart are the middle points of adjacent sides of an unknown rectangle Construct the loci of its vertices

6 Prove that if two similar triangles have their corresponding sides parallel the lines joining corresponding vertices meet in a point

In a given triangle $ABC$ show to inscribe a square $PQRS$ so that $P$ lies on $AB$ $Q$ on $AC$, $R$ and $S$ on $BC$

If $x$ is the length of the side of this square and $h$ the length of the perpendicular $AD$ to the side $BC$, prove that $\frac{1}{x} - \frac{1}{h} = \frac{1}{BC}$

## No. 69

1. Two equilateral triangles $ABC$, $CDE$ are so placed that $D$ is inside the triangle $ABC$ and $E$ on the opposite side of $CA$. Prove that $AE = BD$.

2. If a parallelogram and triangle are on the same base and between the same parallels, state, and prove, the relation between their areas.

On the side $AB$ of a triangle $ABC$ the square $ABDE$ is described. If the area of the triangle $DBC$ is equal to the area of the square, prove that the triangle $ABC$ is obtuse-angled, and that either the side $AE$ of the square will, produced if necessary, bisect $BC$, or the side $BD$ will cut the side $AC$ in a point of trisection.

3. State the construction for drawing the internal common tangents of two non-intersecting circles.

Calculate their lengths when the radii are 3 in. and 4 in., and the centres are 8 in. apart.

4. If $A$, $B$, $C$ are three points in a plane, prove that $AB^2 + AC^2 + 2AB, AC$ is greater than $BC^2$ unless $A$, $B$, $C$ are collinear and $A$ between $B$ and $C$.

5. Two circles, $ABQX$, $ABPY$, cut at $A$ and $B$. $A$, $P$, $Q$ are in a straight line and $B$, $X$, $Y$ are in a straight line. Prove that $PY$, $QX$ are parallel.

6. $OX$, $OY$ are two lines inclined at an angle 70°. A straight line $AB$, 1 in. long, slides so that $A$ is always on $OX$ and $B$ on $OY$. Construct the locus of the centre of the circumcircle of the triangle $AOB$.

## No. 70

1 If the line $AD$ which bisects the base $BC$ of a triangle $ABC$ also *bisects the angle* $BAC$, *prove that* $AB = AC$

2 On the same base $BC$ and on the same side of it are an isosceles triangle $ABC$ and a triangle $DBC$ equal to it in area Prove that the isosceles triangle has the smaller perimeter

3 Construct a trapezium $ABDC$ in which $AC = 2$ cm, $BD = 3$ cm, $CD = 2\,5$ cm, and $AC$, $BD$ are both perpendicular to $AB$

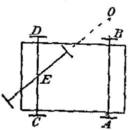
Find, by calculation, the length of $AB$ and the area of $ABCD$

4 Two circles with centres $A$ and $B$, 7 4 cm apart, and of radii 7 0 cm and 2 4 cm respectively, intersect at $P$ and $Q$ Prove that $BP$ and $BQ$ are tangents to the circle with centre $A$

5 If two chords of a circle intersect within the circle, prove that the rectangles contained by their segments are equal

Two chords of a circle, $PA$ and $PB$, are inclined at an angle $60°$, another chord $CD$ cuts $PA$ at $Q$ and $PB$ at $R$ so that $PQ = QR$ Prove that the difference between $PA$ and $PB$ equals the difference between $QC$ and $RD$

6 The figure, which is not drawn to scale shows the plan of four wheels of a cart The axles $AB$ and $CD$ are 4 ft in length and 5 ft 9 in apart In turning, the front axle is turned about, its mid point $E$ through an angle of $8°\ 20$, so that the cart turns about a point $O$ Find by calculation the radii of the circles described by the wheels



## No. 71

1. Draw two non-congruent triangles $ABC$, $XYZ$, having $AB$, $BC$ equal to $XY$, $YZ$, each to each, and the angle $BAC$ equal to the angle $YXZ$. State, and prove, the relation connecting the angles opposite the other pair of equal sides.

2. Prove that parallelograms on the same base and between the same parallels are equal in area. (The proof must be based on the fact that areas which can be made to coincide are equal.)

Construct a trapezium $ABCD$ having $AB = 4{\cdot}5$ cm., $BC = 3{\cdot}6$ cm., $CD = 5$ cm., $DA = 4$ cm., and $AB$ parallel to $DC$. Make a rectangle equal to it in area.

3. By how much does the sum of the squares on the sides containing an acute angle exceed the square on the remaining side of the triangle? Prove your statement when the triangle is obtuse-angled.

4. $ABC$ is a triangle with $C$ obtuse, and $D$ is a point such that $BC$ produced bisects $AD$ at right angles. Prove that $BD$ is a tangent to the circumcircle of the triangle $ABC$ if $BC = CA$; and that if $BD$ cuts the circle at a point $P$ between $B$ and $D$ then $BC$ is greater than $AC$.

5. Prove that the ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

A circle is described about a triangle $ABC$ in which $AB = 6$ in., $BC = 9$ in., $AC = 4{\cdot}5$ in.; the tangent at $A$ meets $BC$ produced at $D$. Prove that $\triangle ABC : \triangle ABD = 7 : 16$.

6. From the top of a vertical cliff, 600 ft. high, the angle of depression of the top of a lighthouse is 10°. The lighthouse is known to be half a mile from the cliff. Find the height of the lighthouse and the angle of elevation of the top of the cliff from the foot of the lighthouse.

## No. 72

1 By using the properties of parallel lines, prove that the sum of the angles of any triangle is two right angles

What other methods of proof are there ?

The sides $AB$, $DC$ of a quadrilateral meet, when produced at $X$ and the sides $DA$, $CB$ meet, when produced, at $Y$, the angles $AXD$, $DYC$ are bisected by lines meeting at $O$ Prove that the angle $XOY$ is equal to half the sum of one pair of opposite angles of the quadrilateral $ABCD$

2 Prove that the three medians of any triangle are concurrent Show that any two of the medians are greater than the third

3 Show that the triangle, having sides 15 cm, 13 cm and 7 cm, is obtuse angled

Calculate the length of the perpendicular let fall from the obtuse angle to the opposite side

4 A point $P$ is taken $2\frac{1}{2}$ in from $O$ the centre of a circle of $1\frac{1}{2}$ in radius Describe a circle of radius 2 in to pass through $P$ and to touch the given circle Show that there are two such circles and measure the distance between their centres

5 If a point $D$ be taken in the base $AC$ of a triangle so that $AD : DC = AB : BC$, prove that $DB$ bisects the angle $ABC$

If $CE$ be drawn perpendicular to $DB$ show that the angles $ACE$, $BCE$ are respectively equal to half the difference and half the sum of the base angles $BAC$ and $BCA$

6 Two circles with radii 3 5 in and 2 7 in are drawn with their centres 8 in apart Calculate the angle between their transverse common tangents

## No. 73

1. From the extremities of a straight line $AB$ are drawn parallel lines $AX$, $BY$, of the same length, but in opposite directions. Prove that $AB$ bisects $XY$.

2. Construct a quadrilateral having each side and one diagonal equal to 1·5 in. Construct a triangle equal to it having one side of length 2·8 in.

3. Prove that any chord of a circle is bisected by the perpendicular let fall on it from the centre.

What is the locus of the middle points (i) of all chords passing through a fixed point, (ii) of all chords parallel to a fixed line.

4. Prove that the opposite angles of a cyclic quadrilateral are supplementary.

The arc $BC$ of the circumcircle of a triangle $ABC$ is bisected at $P$ and $PX$, $PY$ are drawn perpendicular to $AB$, $AC$ respectively. Prove that $BX = CY$.

5. $PQ$ is drawn parallel to the side $BC$ of a triangle $ABC$. Prove that the circumcircles of the triangles $APQ$, $ABC$ touch.

If $PC$ and $QB$ intersect at $O$, prove that the circumcircles of $POQ$ and $BOC$ touch.

6. $AB$ is a diameter of a circle and $OC$ is a radius perpendicular to $AB$; the tangents at $A$ and $B$ meet the tangent at $C$ at $P$ and $Q$ respectively. The other tangent is drawn from $X$, the mid-point of $AP$, and produced to meet $BQ$ at $Y$. Prove that $QY$ is equal to the radius of the circle.

## No. 74

1 Explain clearly what is meant by the converse of a proposition State the converses of the following propositions—

(i) If the angle $ACB$ = the angle $ADB$, then the points $A, B, C, D$ lie on a circle

(ii) If a point $P$ is inside a triangle $ABC$, then the angle $BPC$ is greater than the angle $BAC$

Discuss the truth of these converses

2 Draw a triangle $ABC$ having $BC = 2\ 5$ in, $CA = 1\ 9$ in, $PB = 1\ 4$ in Take a point $D$ in $BC$ so that $BD = 2$ in Now through $D$ a line that will bisect the triangle State and prove your construction

3 State, with proof, how to divide a line into two parts so that the rectangle contained by the whole line and one part is equal to the square on the other part

4 What is the locus of (i) the centres of all circles passing through two given points (ii) the centres of all circles touching two given intersecting straight lines

Being given two finite straight lines $AB$ and $CD$, show how to draw two concentric circles having these lines as chords

5 Two circles with centres $A$ and $B$ cut at $C$, and the angle $ACB$ is a right angle, the line of centres $AB$ cuts the circles between $A$ and $B$ at $D$ and $E$ Prove that the angle $DCE$ is half a right angle

6 If $R$ and $r$ denote the radii of the circumcircle and incircle of a triangle, prove, with or without trigonometry, $R = \frac{abc}{4\Delta}$, $r = \frac{\Delta}{s}$, $R^2 - 2Rr$ = sq of distance between the centres of the circles

[$\Delta$ denotes the area of the triangle $s$ is the semi perimeter ]

## No. 75

1. Draw a line $AB$ of length 1 in.; explain how to find, without using a ruler, a point $C$ in $AB$ produced such that $BC$ is also 1 in. Prove your construction.

2. Give the enunciations of propositions that prove, (i) parallelogram, (ii) triangles, to be equal in area although not equal in all respects.

A point $O$ is taken outside a parallelogram $ABCD$, so that it lies between $AD$ and $BC$ produced. Prove that $\triangle\ AOB = \triangle\ AOC + \triangle\ AOD$.

3. Prove that the diagonals of a parallelogram bisect one another.

Two parallelograms $ABDE$, $BCDE$ are on same base $DE$, and between the parallels $CBA$ and $DE$; $AD$ and $CE$ intersect at $F$. Prove that $BF$, when produced, bisects $DE$.

4. Show how to inscribe a regular polygon of 15 sides in a given circle (i) using a protractor, (ii) not using a protractor.

5. In a trapezium $ABCD$, $AB$ is parallel to $DC$ and $AB = 3$ cm., $BC = 5$ cm., $CD = 9$ cm., $DA = 4$ cm., $DA$ and $CB$ are produced to meet in $O$. Calculate the lengths of $OA$ and $OB$.

Verify by drawing a figure to scale.

6. Using the data of Question 5, calculate the size of the angle $BCD$ and the area of the trapezium.

## No. 76

1 The sides $AB$, $CB$ of a triangle $ABC$ are produced to $D$ and $E$ respectively so that $BD = AB$ and $BE = CB$ Prove that $DE$ is equal and parallel to $AC$

2 Prove from the definition of a parallelogram that a parallelogram is bisected by either diagonal

Construct a parallelogram $ABCD$, having $BC = 2\ 3$ in, angle $ABC = 110°$, and area equal to that of a triangle $DBC$ having angle $DBC = 49°$ and $DB = 2\ 8$ in

3 Prove, without using the proportion properties of parallels, that the straight line joining the mid points of two sides of a triangle is parallel to the third side

If $P$ is the mid point of $AB$, $Q$ of $AC$, and $R$ of $BC$ in a triangle $ABC$, prove that $AR$ bisects $PQ$

4 Construct a triangle $ABC$ having $BC = 3\ 2$ cm, angle $BAC = 100°$ and the median $AD = 1\ 5$ cm

5 If the opposite angles of a quadrilateral are supplementary, prove that the four vertices lie on a circle

The vertex $A$ of a triangle $ABC$ is joined to any point $D$ in the side $BC$ Prove that the centres of the circumcircles of the triangles $ABC$, $ABD$, $ACD$ lie on a circle that passes through $A$

6 Explain how to find the fourth proportional to three given straight lines

By a geometrical construction find the value of $\frac{2\ 3 \times 1\ 8}{3\ 1}$

## No. 77

1. The sides $AB$, $CB$ of a triangle $ABC$ are produced to $D$ and $E$ respectively, so that $BD = BC$ and $BE = BA$. Prove that $AE$ is parallel to $CD$.

2. Prove that two sides of a triangle are greater than the third.

Take two points $A$ and $B$ on the same side of a straight line $XY$; show how to find a point $C$ in $XY$ so that $AC + CB$ is less than $AP + PB$ where $P$ is any other point in $XY$.

3. Prove that the perpendicular to a chord from the centre of a circle bisects the chord.

From a point $P$, 7 cm. from $O$, the centre of a circle with radius 4·5 cm., draw a line so that the chord intercepted on it shall be 5·2 cm. long.

4. If from a point $P$ outside a circle a tangent $PT$ and a secant $PAB$ be drawn, prove that $PT^2 = PA, PB$.

Find, without calculation, a point $P$ in a line $AB$, length 3·5 in., produced so that rectangle $AP \,.\, PB$ may be of area 16 sq. in.

5. A point $O$ is taken on the circumference of a circle and any three chords $OA$, $OB$, $OC$ are drawn; outside the circle a triangle $XYZ$ is drawn having $YZ$ parallel to $OA$, $ZX$, parallel to $OB$, $XY$ parallel to $OC$. Prove $AC : CB = XZ : ZY$.

6. If a polygon with $n$ sides be inscribed in a circle of radius $r$, prove that the perimeter of the circle is $2nr \sin \dfrac{180°}{n}$, and that its area is $\dfrac{n}{2} r^2 \sin \dfrac{360°}{n}$. What are the corresponding values for $n$-sided polygon described about the circle ?

## No 78

1 Define parallel straight lines and from your definition prove that, if a transversal cuts two parallel lines, the alternate angles are equal

Two circles cut at $P$ and $Q$, through $P$ a line $APB$ is drawn cutting one circle at $A$ and the other at $B$, through $Q$ a line $CQD$ is drawn cutting the circle $APQ$ at $C$ and the circle $BPQ$ at $D$ Prove that $AC$ is parallel to $BD$

2 Prove that the diagonals of a rectangle are equal

A line $PQ$ of constant length slides so that $P$ is always on a fixed line $OX$, and $Q$ on another fixed line $OY$ which is perpendicular to $OX$ Determine the locus of $R$ the mid point of $PQ$

3 State the construction for making a triangle equal to a given pentagon $ABCDE$

Construct a regular pentagon with side 1 in, and make an isosceles triangle equal to it (use of protractor is allowed)

4 Prove that angles at the circumference of a circle subtended by the same chord are either equal or supplementary

Through the vertices of a triangle $ABC$ any three lines $YAZ$, $ZBX$, $XCY$ are drawn, forming a triangle $XYZ$ Prove that the circumcircles of the triangle $XBC$, $YCA$, $ZAB$ meet in a point

5 Describe a triangle $ABC$ so that each side touches a given circle of radius 1 7 in, having the angle $A = 42°$, angle $B = 109°$

6 The vertex $A$ of an equilateral triangle $ABC$ is joined to any point $R$ in $BC$, produced, and a point $P$ is taken in $RA$, such that $RP \cdot RC = RB \cdot RA$ The line $BP$ cuts $AC$ at $Q$ Prove that $AB^2 = AQ \cdot BR$

## No. 79

1. Prove that the bisectors of adjacent angles are at right angles.

The centre of the inscribed circle of a triangle $ABC$ is $I$ and $E$ is the centre of the $R$-scribed circle touching $BC$, not produced. Prove that $B$, $I$, $C$, $E$ lie on a circle, the centre of which is on the circumcircle.

2. Construct a quadrilateral which shall be bisected by the diagonal $AC$ and have $AB = 2$ in., $AC = 3$ in., angle $BAC = 32°$, and angle $BCD = 95°$.

3. Enunciate the geometrical theorem corresponding to the algebra formula : $a(a + b) = a^2 + ab$.

If a straight line $AB$ is bisected at $X$ and produced to $Y$, prove that $AX, AY = BX . BY + 2 BX^2$.

4. What are the respective loci of the centres of a circle satisfying the conditions : (i) To touch a given straight line and have a given radius ? (ii) To touch a given circle and have a given radius ?

Two equal circles move so that they touch each other and each touches one of two straight lines $OX$, $OY$, which are at right angles. Find the locus of $R$, their point of contact, (i) by drawing, (ii) by geometrical argument.

5. Prove that the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.

Show that these perpendiculars bisect the angles formed by joining the feet of the perpendiculars.

6. Tangents $CA$, $CB$ are drawn to a circle : any point $P$ is taken on the circle and perpendiculars $PX$, $PY$, $PZ$ drawn to $CA$, $CB$, $AB$ respectively. Prove that $PX . PY = PZ^2$.

## No. 80

1 State the rule for finding the sum of the angles of any rectilineal polygon

$ABCDEFG$ is a regular polygon with 7 sides, calculate the angles of the quadrilateral $CDFG$

2 A straight line $AB$ is bisected at $C$, parallel lines $AX$, $BY$, $CZ$ on the same side of $AB$ meet another straight line at $X$, $Y$ and $Z$ Prove that $AX + BY = 2\,CZ$

What is the similar fact if $AX$, $CZ$ are on the same side of $AB$ and $BY$ on the other side ?

3 Prove Pythagoras's theorem, viz, that the square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides

A perpendicular $ACD$ is drawn to a given line $AB$, $C$ and $D$ being any two points in the perpendicular, $BA$ is produced and in the produced line points $E$ and $F$ are taken so that $AE = BC$ and $AF = BD$ Prove that $DE = CF$

4 Define a tangent to a circle as the limiting position of a secant Deduce from the angle property of a cyclic quadrilateral that an angle between a chord and a tangent at its extremity is equal to the angle in the segment on the other side of the chord

Two circles touch internally at $O$ and a line cuts the larger circle at $P$ and $S$ the inner at $Q$ and $R$ Prove that the angle $POQ =$ the angle $ROS$

Is a similar statement true when the circles touch externally ?

5 Two circles $ABT$, $ABCD$ cut at $AB$ so that $CD$ and $AB$, when produced, meet at $P$, and $PT$ is a tangent to the circle $ABT$ Prove that the circumcircle of the triangle $CDT$ will touch the circle $ABT$

State the construction for drawing a circle to touch a given circle and to pass through two given points

6 Make a triangle $ABC$ having sides $BC = 3\ 4$ cm, $BA = 2$ cm, $CA = 5$ cm Construct an angle $CBD$ such that its sine shall be twice the sine of the angle $BCA$

Measure the angles and check your work by using tables

## No. 81

1. Two triangles $ABC$, $PQR$ have the sides $BC$ and $QR$ in the same straight line and are on the same side of $BR$; also $BC = QR$, $BA = QP$ and angles $ABC$ and $PQR$ are supplementary. Prove that $AP$ is parallel to $BC$ and $QR$.

2. If a parallelogram is held under water, prove that in every position the sum of the depths of the four corners is equal to four times the depth of the intersection of the diagonal.

3. Any point $P$ is taken in the diagonal $AC$ of a parallelogram $ABCD$; through $P$ lines $HPK$, $LPM$ are drawn parallel to the sides of the parallelogram. meeting $AB$, $BC$, $CD$, $DA$ in $M$, $K$, $L$, $H$ respectively. Prove that the parallelograms $PMBK$, $PLDM$ are equal in area.

Hence construct a rectangle with one side 2·3 in. long, and of area 5 sq. in. Verify by measurement.

4. Explain how to draw common tangents to two given circles. State when there are 4, 3, 2, 1, 0 common tangents respectively.

Two circle centres $A$ and $B$ have radii 3 cm. and 2 cm. respectively, and $AB = 7$ cm. Draw a line $PQRS$ cutting circle $A$ at $P$ and $Q$, circle $B$ at $R$ and $S$, so that $PQ = 5$ cm., $RS = 3$ cm.

5. Any point $P$ is taken on the circumcircle of a triangle $ABC$, and perpendiculars $PH$, $PK$, $PL$ are drawn to the sides $BC$, $CA$, $AB$, produced when necessary. Prove that $H$, $K$, $L$ are in the same straight line.

6. The dimensions of a rectangular room are length 20 ft., width 15 ft., height 12 ft. Find, by drawing and calculation. the length of a diagonal and the angle it makes with a diagonal of the floor.

## No. 82

1 $ABC$ is any triangle, show that any number of equilateral triangles can be drawn having their three sides passing respectively through $A$, $B$ and $C$

2 Prove that the opposite sides and angles of a parallelogram are equal

Prove that the figure formed by the intersection of the bisector of the angles of a parallelogram is a rectangle

3 What is the complete locus of points equidistant from two given intersecting straight lines? Prove your answer

Draw two lines $POQ$, $ROS$ intersecting at an angle of 50° Find the locus of a point which moves so that its distance from $POQ$ is always 2 cm greater than its distance from $ROS$

4 Divide a straight line into two parts so that the square on one part may be equal to the rectangle contained by the whole line and the other part

Divide a straight line $AB$, 3 in long, at a point $C$ so that $AB^2 + BC^2 = 3\,AC^2$

5 If $K$ is the orthocentre of a triangle $ABC$, prove that $AK = BC \cot A$

$A$ is a fixed point on the circumference of a circle, and $PQ$ is a variable chord of constant length What is the locus of the orthocentre of the triangle $APQ$?

6 The side $BA$ of a triangle $ABC$ is produced and the exterior angle so formed is bisected by a line which cuts the circumcircle at $D$ and $BC$ produced at $E$ Prove that $AB \cdot AC = AD \cdot AE$

## No. 83

1. Draw two triangles $ABC$, $PQR$, having two sides $AB$, $AC$ equal respectively to $PQ$, $PR$, having also one angle of $ABC$ equal to one angle of $PQR$, and yet the triangles are not congruent.

Two lines meet at $A$. With centre $A$ an arc is drawn cutting the one line at $P$, the other at $Q$, and with centre $A$, another arc is drawn cutting the former line at $X$ and the latter at $Y$. If $PY$, $QX$ intersect at $Z$, prove that $AZ$ bisects the angle at $A$.

2. If a point $O$ is taken inside a triangle $ABC$, prove that the angle $BOC$ is greater than the angle $BAC$.

Show that the converse is not necessarily true.

3. Illustrate by geometrical figures the algebraic formula—

(i) $a^2 - b^2 = (a + b)(a - b)$;
(ii) $(a + b)^2 - (a - b)^2 = 4ab$.

4. Prove that every parallelogram inscribed in a circle must be a rectangle.

If the middle points of the sides of a quadrilateral lie on a circle, prove that the diagonals are at right angles.

5. If two triangles have an angle of the one equal to an angle of the other, and the sides about those angles proportional, prove that the triangles are similar.

In one of the sides $AB$ of a triangle $ABC$, having $AB = AC$, a point $D$ is taken such that $CB = CD$. Prove that $AB . BD = BC^2$.

6. When the sun is at altitude 50° at noon, find (i) the length of the shadow of a post 46 ft. high, (ii) the area of the shadow of a wall, the length of which is 90 ft., the height 7 ft., and the direction north-east.

## No. 84

1 If two sides of a triangle are equal, prove that the opposite angles are also equal

If the point $P$ in the side $BC$ of a triangle which is equi distant from $A$ and $B$ is also the mid point of $BC$, prove that the triangle is right-angled

2 In what kinds of parallelogram (i) are the diagonals at right angles? (ii) do the diagonals bisect the angles?

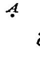
In a trapezium $ABCD$ the non parallel sides $AD$ and $BC$ are equal Prove that $A, B, C, D$ lie on a circle

3 Divide a straight line $AB$, 2·7 in long, at a point $P$, so that $AB^2 = BP^2 + 2\,AB, BP$

4 Prove that the line joining the centre of a circle to the middle point of a chord is perpendicular to the chord

Two circles cut at $X$, $PXQ$ is a line through $X$, meeting one circle at $P$ and the other at $Q$ Prove that $PQ$ is greatest when it is parallel to the line of centres

5 If $I$ is the centre of the inscribed circle of a triangle, prove that the angle $BIC = 90 + \frac{A}{2}$




Describe a triangle $ABC$ having $A$ as one vertex, $P$ the point where the bisector of the angle $B$ meets $AC$, and $Q$ the point where the bisector of the angle $C$ meets $AB$

6 Tangents $PA$, $PB$ are drawn to a circle with centre $O$, and any point $X$ is taken on $AB$ produced The line at right angles to $OX$ at $X$ meets $AP$, $PB$, produced at $Y$ and $Z$ respectively Prove that $XY = XZ$

## No. 85

1. Construct a trapezium $ABCD$ having $AB$ parallel to $DC$ and $AB = 3{\cdot}7$ cm., $BC = 2{\cdot}5$ cm., $CD = 5{\cdot}6$ cm., angle $CDA = 70°$. Take the necessary measurement and find its area.

2. Prove that two triangles are equal in area if they have two sides of the one equal to two sides of the other, each to each, and the included angles together equal to two right angles.

3. Give, without proof, a geometrical illustration of the algebraical identity $(a-b)^2 = a^2 + b^2 - 2ab$.
If one angle of a triangle is one-third of two right angles, prove that the square on the opposite side is less than the sum of the squares on the sides containing that angle, by the rectangle contained by these two sides.

4. Show that two tangents can be drawn to a circle from an external point and that the parts of them intercepted between the point and the circle are equal.
If the inscribed circle of the triangle $ABC$ touches $AB$ at $R$, prove that $2\,AR = AB + AC - BC$.

5. Prove that equal chords of a circle subtend equal angles at the centre, and that the angles they subtend at a point on the circumference are either equal or supplementary.
Two equal circles $ABC$ and $ADC$ cut at $A$ and $B$, and $AB = AC$. If $BC$ meets the other circle at $D$, prove that $D$ is inside the circle $ABC$.

6. If $I$ is the centre of the inscribed circle of a triangle $ABC$, prove that $AI : ID = BA + AC : BC$ where $D$ is the point of intersection of $AI$ and $BC$.

## No. 86

1 State and prove a construction for drawing a line perpendicular to a given line from a given point outside it

$A$ and $B$ are given points on opposite sides of a given line $XY$ Find a point $C$ in $XY$ such that the angle $ACX$ is equal to the angle $BCX$.

2 Prove that the quadrilateral formed by joining the middle points of the sides of a rectangle is a rhombus, and that its area is half that of the rectangle

3 If $P$ and $Q$ are points on the same side of $AB$ such that the angles $APB$, $AQB$ are equal, prove that the four points $A$, $P$, $Q$, $B$ lie on a circle

Equilateral triangles $APB$, $AQC$ are described on the sides $AB$, $AC$ of a triangle $ABC$, falling outside that triangle If $BQ$, $CP$ intersect at $R$, prove that $A$, $C$, $Q$ $R$ lie on a circle

4 In any triangle prove that the difference of the squares on two sides is equal to the rectangle contained by the third side, and the projection on it of the median bisecting it

If $A$, $B$, $C$, $D$ are four points such that $AC^2 - BC^2 = AD^2 - BD^2$, prove that $CD$ is perpendicular to $AB$

5 Explain how to inscribe a circle in a given triangle Show that the radius is equal to the area of the triangle divided by the semi perimeter

6 Tangents $PA$, $PB$ are drawn from a point $P$ to a circle with centre $O$, and $PO$ cuts $AB$ at $C$ Prove that $OC$ $CP$ $= AC^2$ If $XY$ is any other chord through $C$, prove that $OP$ bisects the angle $XPY$

## No. 87

1. If two triangles have their sides equal, each to each, prove that they are equal in all respects.

On the side $AB$ of a square $ABCD$ an equilateral triangle $ABE$ is described, and $F$ is the mid-point of $CD$. Prove that $EF$ is at right angles to $CD$.

2. Construct a quadrilateral $ABCD$ having $AB = 1{\cdot}7$ in., diagonal $AC = 2{\cdot}3$ in., angle $BAC = 80°$, angle $BCD = 90°$, angle $ADC = 90°$. Make an isosceles triangle equal to the quadrilateral, having $AC$ as base.

3. A cone stands on a base of radius 5 in. and its slant side is of length 13 in.; find, by drawing or calculation, the radius of the largest sphere that can be covered by the cone.

4. In a triangle $ABC$ the angle at $B$ is obtuse, $AD$ is let fall perpendicular to $BC$, meeting it at $D$. Prove that $D$ must be outside the triangle, and that $B$ is between $C$ and $D$. Prove $AB^2 = AC^2 + BC^2 - 2\,BC\,.\,CD$.

5. If a straight line $PQ$ cuts the sides $AB$, $AC$ of a triangle in the same ratio, prove that $PQ$ is parallel to $BC$.

Take any point $P$ in the side $AB$ of a triangle $ABC$, and show how to draw a line through $P$ to meet $BC$ produced at $Q$, such $PQ$ may be bisected by $AC$.

6. A regular pyramid has a square base; the faces are isosceles triangles with base angles each equal to 70°. Find the inclination of the faces to the base (i) by drawing, (ii) by trigonometrical calculation.

## No 88

1 Make an angle $XOY$ equal to $55°$, on $OX$ measure $OA = 3$ cm, $OB = 7$ cm From $A$ and $B$ draw two equal straight lines $AC$ and $BC$ to meet on $OY$ Measure them

2 If the intercepts made by three parallel lines on any one transversal are equal prove that the intercepts made on any other transversal are also equal

Perpendiculars $BP$, $CQ$ are let fall from the vertices $B$ and $C$ of a triangle $ABC$ upon any line through $A$ between $AB$ and $AC$ Prove that $D$, the mid point of $BC$, is equidistant from $P$ and $Q$

3 Explain how to describe an equilateral triangle about a given circle

$ABC$ is an equilateral triangle described about a circle with centre $O$, and $D$ is the point of contact of $BC$ with the circle If $AD$ cuts the circle at $E$ prove that $AE = EO = OD$

4 Divide a straight line $AB$ of length 7 cm into two parts at $C$ so that the rectangle $AC$ $CB$ may be of area 9 sq cm

Solve the equation $x^2 - 7x + 9 = 0$ by a geometrical construction

5 $ABC$ is an equilateral triangle inscribed in a circle, $D$ is a point on the circumference in the minor arc $BC$ On $BD$ an equilateral triangle $BED$ is described, so that $E$ falls inside the circle Prove that $A$, $E$, $D$ are in same straight line

6 With the data in Question 1, find the lengths of $AC$ and $BC$ with the aid of trigonometry

## No. 89

1. Two triangles $ABC$, $DBC$ on the same side of the base $BC$ have $AB = DC$ and $AC = DB$; $AC$ and $DB$ intersect in $E$. Prove that $EB = EC$.

2. Prove that, if a transversal cuts two parallel lines, the two interior angles on the same side are supplementary.

The bisectors of the angles $B$ and $C$ of a parallelogram $ABCD$ meets at a point $P$, and the length of $AB$ is 6 in. If the point $P$ lies in the side $AD$, find the length of $BC$.

3. Construct a triangle $ABC$ having $AB = 3$ in., and the angles $A, B, C$ in the ratio $2 : 3 : 4$. Make a square equal to it without any further calculation. Measure the side of the square to the nearest fortieth of an inch.

4. At a point $P$ on a circle of radius 2 in., a tangent $PQ$ is drawn of length 3 in. Describe a circle passing through $Q$, touching the first circle and having its centre on $PQ$.

5. The side $AC$ of a triangle $ABC$ is produced both ways to $D$ and $E$, so that $AD = AB$ and $CE = CB$, and a circle with centre $O$ is described to pass through $B$, $D$, and $E$. Prove that $OB$ bisects the angle $ABC$.

6. Prove Ptolemy's theorem, viz. the sum of the rectangles contained by the opposite sides of a cyclic quadrilateral is equal to the rectangle contained by the diagonals.

An equilateral triangle $ABC$ is inscribed in a circle; $P$ is a point on the minor arc $BC$. Prove that $PA = PB + PC$.

## No. 90

1. Prove that, in any triangle, the side opposite the greater of two of the angles is greater than the side opposite the less.

Points $P$ and $Q$ are taken on the sides $AB$, $AC$ respectively of a triangle $ABC$. If the angle $A$ is the greatest angle, prove that $PQ$ must be less than $BC$.

2. Describe a triangle $ABC$, being given that $BC = 4{\cdot}7$ cm., angle $BCA = 79°$, and that the radius of the circumcircle is 3 cm. Prove your construction to be correct.

3. On the same side of $AB$ an equilateral triangle $APB$ and a square $ABCD$ are described Prove that $P$ falls inside the square.

If $BP$ produced meets $CD$ at $Q$, prove that the angle $DPQ$ is half a right angle.

4. Two equal circles have centres $A$ and $B$; a line parallel to $AB$ cuts the $A$ circle at $P$ and $Q$, and the $B$ circle at $H$ and $K$, so that $PQ$ and $HK$ are in the same sense. Prove that $PH = QK = AB$.

Show how to draw a circle to touch two given parallel straight lines and to pass through a given point between them.

5. Prove that the bisector of the vertical angle of a triangle divides the opposite side into segments which have the same ratio as the sides containing the bisected angle.

$AD$ is a median of the triangle $ABC$, the angles $ADB$, $ADC$ are bisected by lines meeting $AB$, $AC$, at $E$ and $F$ respectively. Prove that $EF$ is parallel to $BC$.

6. $PQ$ and $XY$ are parallel lines 4 cm. apart, and $A$ is a point between them 1·5 cm. from $XY$. Construct a square $ABCD$ having $B$ on $XY$ and $D$ on $PQ$; prove the accuracy of the construction. Measure the angle $ABX$ and verify by trigonometry.

## No. 91

1. In a triangle $ABC$ the sides $AB$ and $AC$ are equal; the angles $ABC$, $ACB$ are bisected by straight lines meeting at $D$. Prove that $AD$ produced bisects $BC$.

2. Prove that two sides of any triangle are greater than the third side.

Prove that any two medians of a triangle are greater than the third, and that the sum of the medians is less than the perimeter of the triangle.

3. Take a point $O$ 3 cm., from a line $XY$. Draw the locus of the middle points of lines drawn from $O$ to $XY$. Prove the truth of your construction.

4. Prove, by a geometrical figure, that the difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.

A point $O$ is taken in the base $BC$ of an isosceles triangle $ABC$; prove that $AB^2 - AO^2 = BO \,.\, OC$.

5. Two circles intersect at $A$; a line $PAQ$ meets one of the circles at $P$, the other at $Q$, and bisects an angle between the tangents at $A$. Prove that $PA : AQ$ equals the ratio of the radii.

6. Prove that the area of any triangle is equal to the rectangle contained by two sides multiplied by the sine of the included angle.

$A$, $B$, $C$ are three fixed points in order on a straight line and $O$ is any point outside the line. Prove that:

(i) $OB \sin AOB : OC \sin AOC$ is the same for all positions of $O$

(ii) $\dfrac{\sin AOB}{OC} + \dfrac{\sin BOC}{OA} = \dfrac{\sin COA}{OB}$.

## No 92

1 If two parallelograms have two adjacent sides of the one equal to two adjacent sides of the other, each to each and one angle of the one equal to one angle of the other prove that the parallelograms will coincide if one is superposed on the other

2 Draw a triangle $ABC$ having $BC = 3\ 2$ cm $CA = 2\ 1$ cm $AB = 3$ cm On a base $PQ$ of length 2 in construct without calculation a triangle $PQR$ equal in area to the triangle $ABC$

3 A triangle $PBC$ is drawn on a given line $BC$ 2 1 in long Find the locus of $P$ if $PB^2 - PC^2 = \ 7$ of a sq in

4 Prove that if two circles touch, either internally or externally the point of contact and the two centres are in the same straight line

Circles are drawn to touch a given circle and a given straight line Show that part of the complete locus of their centres is the same as the locus of a point equidistant from a certain fixed point and from a certain fixed line

5 Two circles intersect at $A$ show how to draw a straight line through $A$ so that the circles shall intercept equal chords on that line

Show how to draw a square so that opposite sides may pass through two given points $A$ and $B$ and the diagonals may intersect at another given point $C$

6 The tangent at $A$ to the circumcircle of a triangle $ABC$ meets $BC$ produced at $P$ Prove that—

$$PB \quad PC = AB^2 \quad AC^2$$

## No. 93

1. $ABC$ is a triangle, right-angled at $B$; $AB = 1{\cdot}7$ in., and angle $A$ is 57°. Draw the triangle and measure $AC$, $BC$.

Confirm your measurements by calculating these lengths by using trigonometrical tables.

2. Prove that the opposite sides and angles of a parallelogram are equal.

The vertex $A$ of a parallelogram $ABCD$ is joined to any point $E$ in the side $CD$; any point $F$ is taken in $EA$, and $EA$ is produced to $G$ so that $AG = EF$. Prove that the parallelogram $FGHB$ is equal in area to the parallelogram $ABCD$.

3. If there are two straight lines, one of which is divided into two parts, show by a figure that the rectangle contained by the undivided line and a part of the divided line is equal to the rectangle contained by the two lines diminished by the rectangle contained by the undivided line and the other part of the divided line.

Hence give a geometrical demonstration of the algebraical identity $a(b-c) + b(c-a) = c(b-a)$.

4. In a triangle $ABC$, the perpendiculars from $B$ and $C$ on the opposite sides intersect at $D$, and meet the circumcircle in $E$ and $F$. Prove that $A$ is the centre of the circumcircle of the triangle $DEF$.

5. Prove that the locus of a point whose distances from two given points are in a constant ratio is a circle.

6. From a point $P$ in the side $BC$ of a triangle $ABC$, $PQ$ is drawn parallel to $AB$ to meet $AC$ in $Q$, and $PR$ parallel to $AC$ to meet $AB$ in $R$, and $QR$ is produced to meet $BC$ at $S$. Prove that $SP$ is the mean proportional between $SB$ and $SC$.

## No. 94

1 Two lines $OA$ and $OB$ are respectively at right angles to $OX$ and $OY$, determine whether the bisector of an angle between $OA$ and $OB$ also bisects one of the angles between $OX$ and $OY$

2 The diagonals of a parallelogram $ABCD$ intersect at $O$, and a line $POQ$ cuts $AB$ at $P$ and $DC$ at $Q$, prove that $PQ$ is bisected at $O$

Draw a triangle $ABC$ having $BC = 2\ 3$ cm, $CA = 1\ 5$ cm, $AB = 1\ 7$ cm Construct a rhombus so that two adjacent sides contain an angle 60° and pass through $A$ and $B$ respectively, and the diagonals intersect at $C$ Prove your construction

3 Show how to produce a given straight line so that the rectangle contained by the whole line thus produced, and the part produced, is equal to the square on the original line

4 Prove that an angle at the centre of a circle is double any angle at the circumference standing on the same arc

Two circles, $PQX$ and $PQY$, cut orthogonally (i e their tangents at a point of intersection are at right angles) at $P$ and $Q$, $PX$ and $PY$ are two chords at right angles Prove that $X$, $Q$, $Y$ are in the same straight line

5 If two triangles are equiangular, prove that their corresponding sides are in the same ratio

In Question 4, prove that the ratio of $PX$ to $QY$ is equal to the ratio of the diameters of the circles

6 An observer, whose eye $C$ is 5 6 ft from the ground and 90 ft from a vertical tower $AB$, finds that the tower subtends at his eye an angle of 59° Calculate the height of the tower

## No. 95

1. $ABCD$ is a square, with $A$ as centre a circle is described cutting $AB$ at $E$ and $AD$ at $F$. Prove that the line through $A$ at right angles to $DE$ bisects $BF$.

2. $ACB$ is an isosceles right-angled triangle with $AC$ equal to $BC$; at any point $P$ in $AB$, nearer to $B$ than $A$, a perpendicular is erected meeting $BC$ at $Q$. Prove that $PQ = PB$.

Hence show how to divide a given line $AB$ so that the sum of the squares on the two parts may be as small as possible.

3. Deduce from the construction of Question 2 that, if a straight line be divided into two equal parts and also into two unequal parts, the sum of the squares on the two unequal parts is equal to twice the sum of the squares on half the line and the line between the points of section.

4. Prove that angles in the same segment of a circle are equal to one another.

If the segment is smaller than a semi-circle, prove this proposition without using reflex angles.

5. Give a geometrical proof of the proposition that if four straight lines are in proportion, the rectangle contained by the means is equal to the rectangle contained by the extremes.

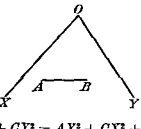
Prove that either of the equal sides of an isosceles triangle is a mean proportional between the diameter of the circumscribing circle, and the perpendicular from the vertex.

6. Describe a regular pentagon with a side of 1 in., the use of protractor being allowed. State how it might be done without using a protractor. Calculate the distance of a vertex from the opposite side.

Hence, or otherwise, show that if a pentagon and a hexagon are described on the same side of the same base, the pentagon will fall entirely inside the hexagon.

## No. 96

1 If a straight line cuts two other straight lines, which are not parallel prove that the alternate angles are unequal, the smaller being on the side towards which the lines tend to meet (The properties of parallel lines are not to be assumed )




2 Explain, with proof, how to draw a parallelogram $ABCD$, having $C$ on $OY$ and $D$ on $OX$

3 If from a point $O$ within a triangle $ABC$, perpendiculars $OX$, $OY$, $OZ$ are let fall on $BC$, $CA$, $AB$ respectively, prove that $AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$

Where must $O$ be taken so that $AZ^2 + ZB^2 + BX^2 + XC^2 + CY^2 + YA^2$ may be as small as possible ?

4 Construct a triangle $ABC$ being given $BC = 2$ in , angle $BAC = 130°$, and the median $AD = 7$ in

5 If a point $O$ is taken inside a triangle $ABC$, and $AO$, $BO$, $CO$ are produced to meet the opposite sides at $P$, $Q$ $R$ respect ively, prove that $\frac{BP}{PC}\ \frac{CQ}{QA}\ \frac{AR}{RB} = 1$ [This is Ceva's Theorem ]

6 A circle passing through $A$, a vertex of the rectangle $ABCD$, cuts $AB$ at $P$, $AC$ at $Q$ $AD$ at $R$ prove that—

(i) $AB\ AP + AD\ AR = AC\ AQ$

(ii) $AB\ BP + AD\ DR = AC\ CQ$

## No. 97

1. Prove that the triangle formed by joining the mid-points of the sides of an isosceles triangle is also isosceles.

2. Construct a rectangle having two of the sides each 1·4 in. long, and a diagonal 3·1 in. long. Measure the other sides to the nearest tenth of an inch.

3. $A$ and $B$ are two points on the same side of a line $XY$; show how to find a point $L$ in $XY$ such that $LA + LB$ is a minimum.

Prove that the triangle of minimum perimeter that can be inscribed in a given triangle is formed by joining the feet of the perpendiculars let fall from the vertices on the opposite sides.

4. The bisector of the angle $A$ of a triangle $ABC$ cuts the perpendicular bisector of the side $BC$ at $D$. Prove that $A$, $B$, $C$, $D$ are concyclic.

If $P$ is any point not on the circumcircle of the triangle $ABC$, prove that the line joining the centres of the circumcircles of the triangles $ADP$ and $BCD$ bisects $AD$.

5. Show that four circles can, in general, be found to touch each of three given straight lines; and prove that the centre of any one of these circles is the orthocentre of the triangle formed by the centres of the other three circles.

6. Prove that if two triangles are equiangular, their areas are proportional to the squares on corresponding sides.

Construct (i) a square of area 15 sq. in., (ii) a square equal in area to an equilateral triangle of side 1 in. Hence, construct an equilateral triangle of area 15 sq. in. Verify by calculation.

## No 98

1 A brass weight consists of a cylindrical portion surmounted by a cone, the top of the cylinder being the base of the cone The radius of the common section is 1 1 in, the total height is 5 in, the slant height of the cone is 3 in Draw the elevation (i e side view) of the brass weight

2 Show how to draw a line from a point $P$ in the side $BC$ of a triangle $ABC$, which will bisect the triangle

3 If the diagonal $AC$ of a rhombus $ABCD$ is produced to any point $P$, prove that $PA\ \ PC = PB^2 - AB^2$

4 Draw a circle of radius 3 cm, and in it inscribe a triangle having angles 33° and 66°

About the same circle describe a triangle, equiangular to the former, the sides of which touch the circle

In each triangle measure the sides opposite the angles 66° Verify by trigonometrical calculation

5 Prove that the rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by pairs of opposite sides

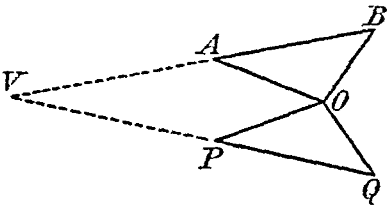
A point $D$ is taken in the side $BC$ produced, of an equilateral triangle $ABC$, and on $AD$, on the opposite side to $C$, an equilateral triangle $ADE$ is drawn Prove that $CE = BD$

6 Prove the formula $c^2 = a^2 + b^2 - 2\,ab \cos C$, considering the three cases (i) $C$ acute, (ii) $C$ obtuse, (iii) $C$ a right angle

Find all the angles of the triangle whose sides are 5 8 cm, 4 2 cm, 4 0 cm

## No. 99

1. Explain the method of proof by *reductio ad absurdum*. Use this method to prove that if two angles of a triangle are equal, the opposite sides are also equal.




$OAB$, $OPQ$ are two triangles having $OA = OP$, $OB = OQ$ and $AB = PQ$; $BA$ and $QP$ are produced to meet at $V$ as in the figure. Prove that $VB = VQ$.

2. Without using any proposition about areas other than the equality of congruent triangles, prove that parallelograms on the same base and between the same parallels are equal in area.

Make a parallelogram $ABCD$ having $AB = 2$ in., $BC = 3$ in., and the diagonal $BD = 4$ in. Construct a parallelogram equal in area to $ABCD$, having two sides each equal to 2·5 in., and two angles each equal to $110°$.

3. State a necessary condition involving angles about four points that lie on a circle.

In a triangle $ABC$, $AP$, $BQ$ are perpendiculars from $A$ and $B$ on the opposite sides, $E$ and $F$ are the mid-points of $AC$ and $AB$ respectively. Prove that $P$, $Q$, $E$, $F$ lie on a circle.

4. $A$ and $B$, the centres of two circles of radii 2·3 cm. and 3 cm. respectively, are 6 cm. apart. Find all the points $P$ such that the tangents from $P$ to the $A$ circle are 3 cm. long, and those to the $B$ circle are 4 cm. long.

5. With centre $C$ on the circumference of a given circle another circle is described cutting the former at $A$ and $B$; through $A$ a line is drawn cutting the first circle at $P$, and the second at $Q$. Prove that $PB = PQ$.

6. Four circles can be described each touching the sides or sides produced of a triangle. Prove that the circumcircles of the four triangles formed by the centres of these circles are equal.

## No. 100

1 Prove that an exterior angle of a triangle is equal to the sum of the interior opposite angles

Find a point $X$ in the side $BC$ of a triangle $ABC$, such that $XD$, the perpendicular on $AB$, is equal to $XC$ State and prove your construction

2 The diagonal $AC$ of a quadrilateral $ABCD$ is produced through $C$ to any point $E$ If $AB = AD$ and $CB = CD$, prove that $EA$ bisects the angle $BED$

3 Prove that in equal circles (or the same circle) equal angles at the circumferences stand on equal chords

A point $P$ moves so that the angle $APB$ is constant in size, $AB$ being fixed in position and magnitude In any position of $P$, $AK$ is let fall perpendicular to $BP$ and $BL$ perpendicular to $AP$ Prove that $KL$ is of constant length

4 Find a point $P$ on the circumcircle of a triangle $ABC$, so that, if $AP$ is produced to meet $BC$ produced at $Q$, $AQ$ may be bisected at $P$ Prove the truth of your construction

5 If at a point $A$ on a circle a chord $AB$ is drawn, and another line $AX$ is drawn on the side of the minor segment, such that the angle $XAB$ is equal to the angle in the major segment, prove that $AX$ is a tangent to the circle

$AB$, $BC$, $CD$ are three equal straight lines such that $BA$ and $CD$ are parallel on opposite sides of $BC$ The circumcircle of the triangle $ABC$ cuts $CD$ at $E$ Prove that the rectangle $AB$, $DE$ is equal to the square on $AC$

6 If two triangles have an angle of the one equal to an angle of the other, and the sides about those angles proportional, prove that the triangles are similar

$ACB$, $ADB$ are two right angled triangles on the same side of the hypotenuse $AB$, and $AC = BD = \frac{1}{4}AB$ $O$ is the mid point of $AB$, and $AC$ is produced to $E$, making $AE$ equal to $AO$ If $BE$ cuts $OD$ at $F$, prove that $OF = BF$

MADE IN GREAT BRITAIN AT THE PITMAN PRESS BATH
CS—(E 119